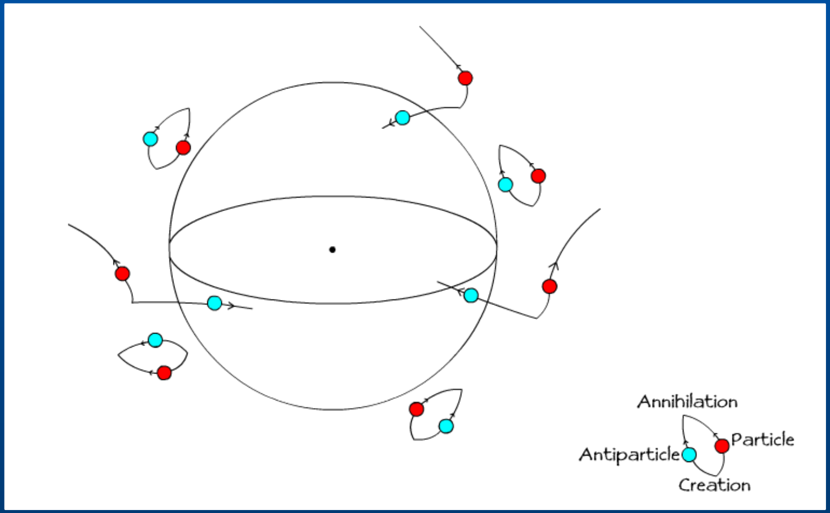
GARRA Group

Black hole astrophysics



Gustavo E. Romero
Grupo de Astrofísica Relativista y Radioastronomía
Instituto Argentino de Radioastronomía, CONICET-CIC-UNLP
Facultad de Ciencias Astronómicas y Geofísicas, UNLP
romero@fcaglp.unlp.edu.ar

Curso 2022

Orbits around Schwarzschild black holes

In general, if one knows the metric, then one can calculate any geodesic trajectory in space-time

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0.$$

Orbits around a Schwarzschild black hole can be easily calculated using the metric and the relevant symmetries.

Let us call k^μ a vector in the direction of a given symmetry (i.e. k^μ is a Killing vector). A static situation is symmetric in the time direction, hence we can write $k^\mu = (1, 0, 0, 0)$.

The 4-velocity of a particle with trajectory $x^\mu = x^\mu(\tau)$ is $u^\mu = dx^\mu/d\tau$. Then, since $u^0 = E/c$, where E is the energy, we have:

$$g_{\mu\nu}k^{\mu}u^{\nu} = g_{00}k^{0}u^{0} = g_{00}u^{0} = \eta_{00}\frac{E}{c} = \frac{E}{c} = \text{constant.}$$

If a particle moves along a geodesic in Schwarzschild geometry:

$$\boxed{c\left(1 - \frac{2GM}{c^2 r}\right)\frac{dt}{d\tau} = \frac{E}{c}.}$$

Similarly, for the symmetry in the azimuthal angle ϕ we have $k^\mu = (0,0,0,1)$, in such a way that:

$$g_{\mu\nu}k^\mu u^\nu = g_{33}k^3u^3 = g_{33}u^3 = -L = \text{ constant.}$$

In the Schwarzschild metric we find, then,

$$r^2\frac{d\phi}{d\tau} = L = \text{ constant.}$$

Dividing the interval by $c^2\ d\tau^2$

$$1 = \left(1 - \frac{2GM}{c^2r}\right)\left(\frac{dt}{d\tau}\right)^2 - c^{-2}\left(1 - \frac{2GM}{c^2r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 - c^{-2}r^2\left(\frac{d\phi}{d\tau}\right)^2$$

Using the conservation equations

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{c^2} - \left(c^2 + \frac{L^2}{r^2}\right)\left(1 - \frac{2GM}{c^2r}\right).$$

Expressing the energy in units of mc^2 and introducing an effective potential V_{eff},

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{c^2} - V_{\text{eff}}^2.$$

For circular orbits of a massive particle we have the conditions

$$\frac{dr}{d\tau} = 0 \quad \text{and} \quad \frac{d^2r}{d\tau^2} = 0.$$

The orbits are possible only at the turning points of the effective potential:

$$V_{\text{eff}} = \sqrt{\left(c^2 + \frac{L^2}{r^2}\right)\left(1 - \frac{2r_{\text{g}}}{r}\right)},$$

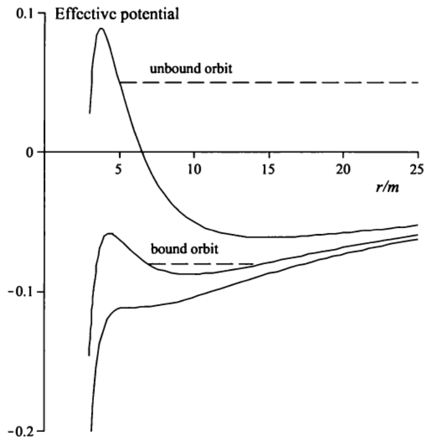
L is the angular momentum in units of mc and $r_g = GM/c^2$ is the gravitational radius. Then,

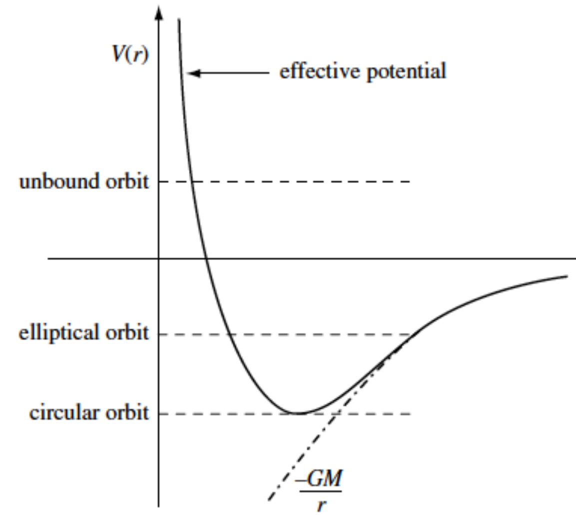
$$r = \frac{L^2}{2cr_g} \pm \frac{1}{2}\sqrt{\frac{L^4}{c^2 r_g^2} - 12L^2}.$$

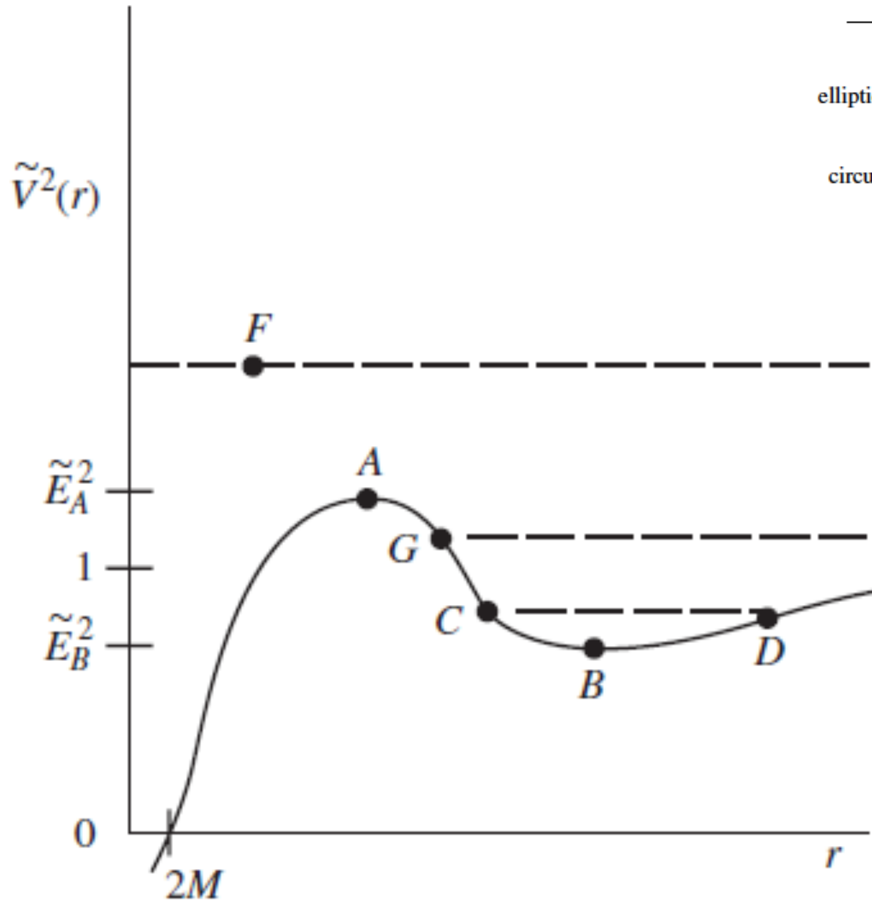
Orbits around Schwarzschild black holes

For $L^2 > 12c^2r_g^2$ there are two solutions. The negative sign corresponds to a maximum of the potential and is unstable. The positive sign corresponds to a minimum, which is, consequently, stable.

At $L^2 = 12c^2r_g^2$ there is a single stable orbit. It is the innermost marginally stable orbit, and it occurs at $r = 6r_{\mathrm{g}} = 3r_{\mathrm{Schw}}$.



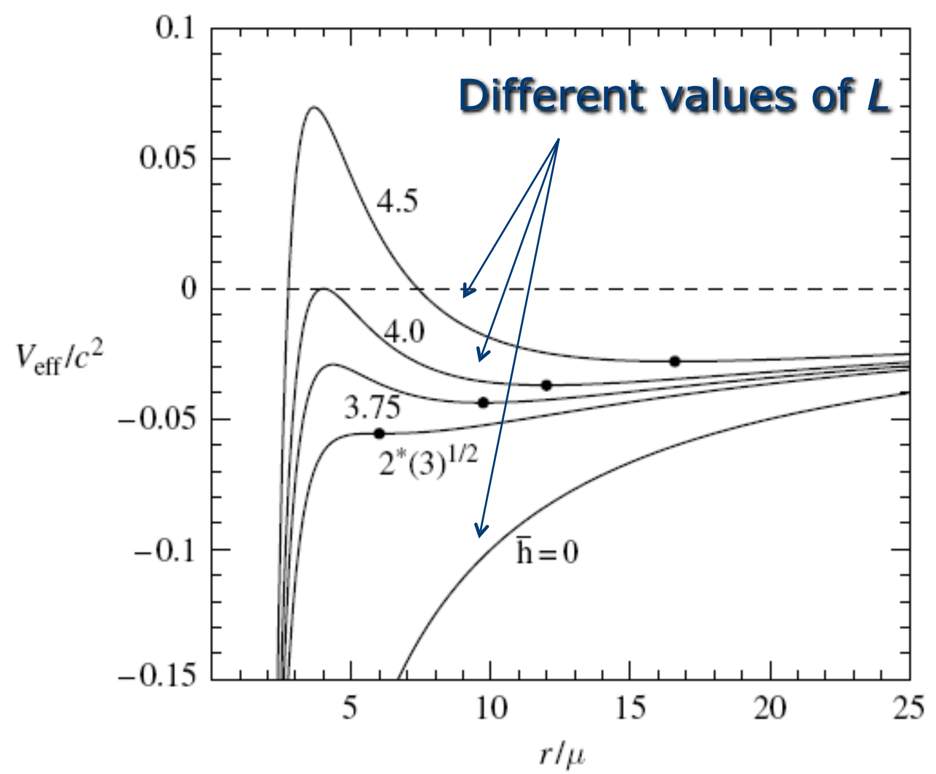




Typical effective potential for a massive particle of fixed specific angular momentum in the Schwarzschild metric.

The effective potential

$$V_{\text{eff}} = \sqrt{\left(c^2 + \frac{L^2}{r^2}\right)\left(1 - \frac{2r_g}{r}\right)},$$



Orbits around Schwarzschild black holes

The specific angular momentum of a particle in a circular orbit at r is:

$$L = c\left(\frac{r_g r}{1 - 3r_g/r}\right)^{1/2}.$$

Its energy (units of mc^2) is:

$$E = \left(1 - \frac{2r_g}{r}\right)\left(1 - \frac{3r_g}{r}\right)^{-1/2}.$$

The proper and observer's periods are:

$$\tau = \frac{2\pi}{c}\left(\frac{r^3}{r_g}\right)^{1/2}\left(1 - \frac{3r_g}{r}\right)^{1/2}$$

and

$$T = \frac{2\pi}{c}\left(\frac{r^3}{r_g}\right)^{1/2}.$$

Orbits around Schwarzschild black holes

When $r \to 3r_g$ both L and E tend to infinity, so only massless particles can orbit at such a radius. Photons orbiting at such a distance form the *photosphere* of the BH. The photosphere is 1. unstable, and 2. unobservable.

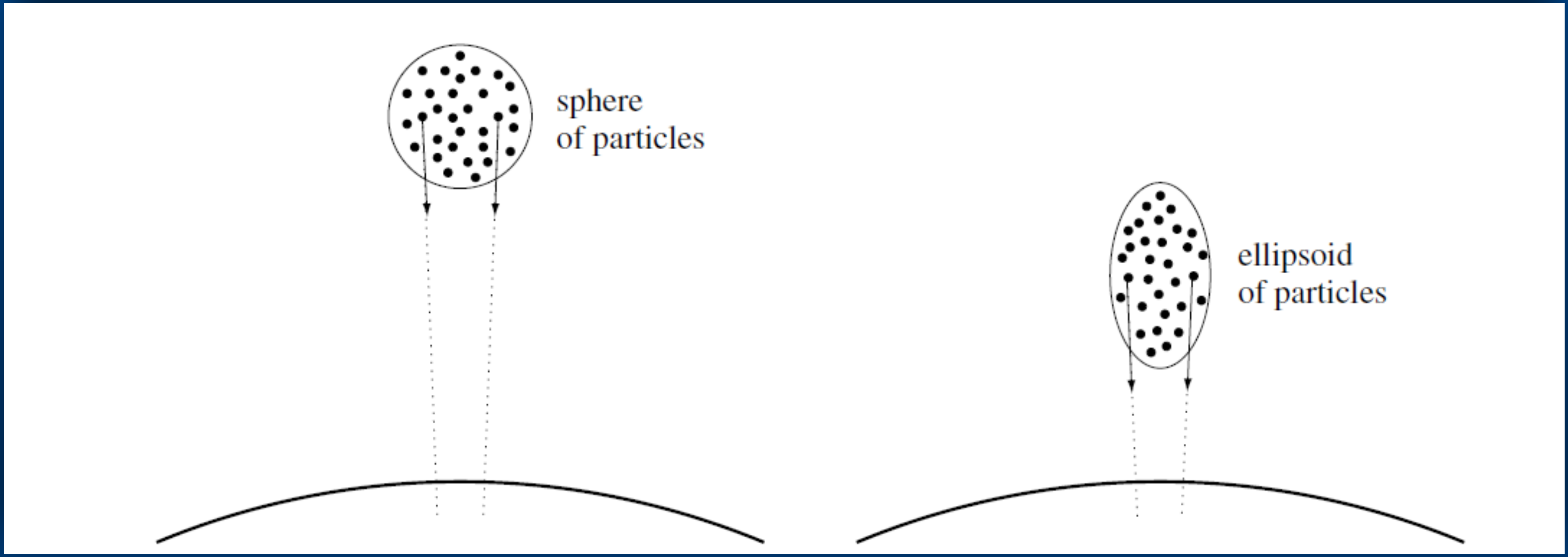
The energy of a particle in the innermost stable orbit can be obtained from the above equation for the energy setting $r = 6r_g$. This yields (units of mc^2):

$$E = \left(1 - \frac{2r_g}{6r_g}\right)\left(1 - \frac{3r_g}{6r_g}\right)^{-1/2} = \frac{2}{3}\sqrt{2}.$$

Since a particle at rest at infinity has $E = 1$, then the energy that the particle should release to fall into the black hole is $1 - (2/3)\sqrt{2} = 0.057$. This means 5.7 % of its rest mass energy, significantly higher than the energy release that can be achieved through nuclear fusion.

Tidal forces

Tidal forces are the effect of spacetime curvature on extended bodies. Different parts of the body experience different accelerations.



Tidal forces

The local velocity at r of an object falling from rest to the black hole is

$$v_{\text{loc}} = \frac{\text{proper distance}}{\text{proper time}} = \frac{dr}{(1 - 2GM/c^2 r)dt}.$$

Hence, using the expression for dr/dt from the metric

$$\frac{dr}{dt} = -c\left(\frac{2GM}{c^2 r}\right)^{1/2}\left(1 - \frac{2GM}{c^2 r}\right),$$

we have,

$$v_{\text{loc}} = \left(\frac{2r_{\text{g}}}{r}\right)^{1/2} \quad \text{(in units of } c\text{)}.$$

Tidal forces

$$v_{\mathrm{loc}} = \left(\frac{2r_{\mathrm{g}}}{r}\right)^{1/2}$$

$$dv_{\mathrm{loc}}/d\tau = (dv_{\mathrm{loc}}/dr)(dr/d\tau) = (dv_{\mathrm{loc}}/dr)v_{\mathrm{loc}} = r_{\mathrm{g}}c^2/r^2.$$

$$\Downarrow$$

$$dg = \frac{2r_{\mathrm{g}}}{r^3}c^2 dr.$$

Tidal forces

The tidal acceleration on a body of finite size Δr is simply

$$\Delta g=(2r_g/r^3)c^2\Delta r.$$

This acceleration and the corresponding force becomes infinite at the singularity. As the object falls into the black hole, tidal forces act to tear it apart.

“Spaghettification”

The differential acceleration that an object will experience along an element *dr* is:

$$dg = \frac{2r_g}{r^3} c^2 dr.$$

Notice that the tidal force will increase as the object approaches from infinity to the black hole. For a solar mass black hole the tidal forces can be deadly even before crossing the event horizon. As the body approaches the singularity the tidal forces tend to infinite.

Surface gravity

We can integrate $$dg = \frac{2m}{r^3} dr.$$ from *2m* to ∞

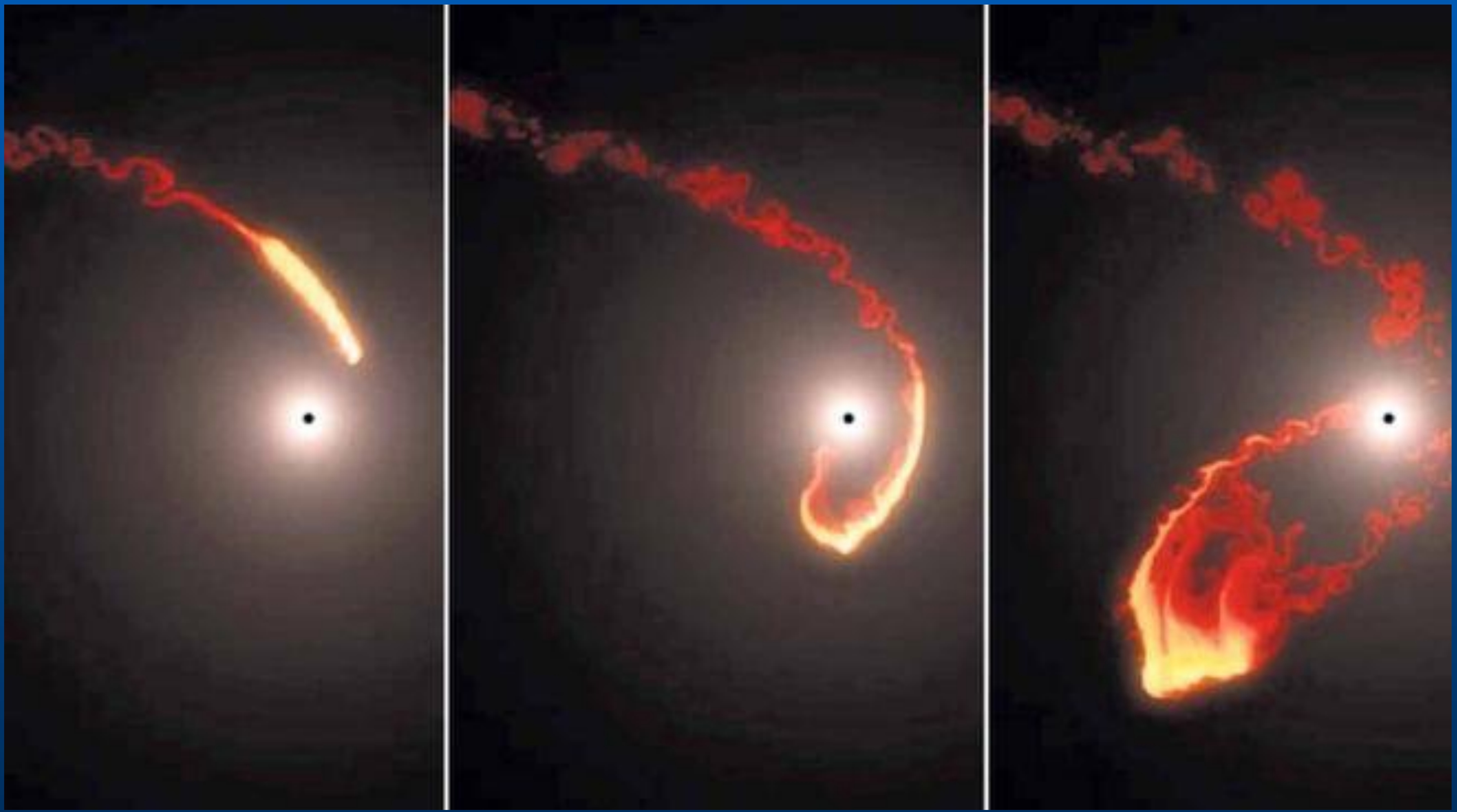
to get $$g_\infty = m/(2m)^2 = 1/4m.$$

Or to evaluate the Newtonian potential (infinite limit) at *2m*:

$$mg_\infty = \frac{GmM}{r_{\text{Schw}}^2} = \frac{m}{4GM}.$$

Effect of tidal forces

Effect of tidal forces



Radial motion of photons

In the case of photons we have that $ds^2 = 0$. The radial motion, then, satisfies:

$$\left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 = 0.$$

$$\frac{dr}{dt} = \pm c\left(1 - \frac{2GM}{rc^2}\right).$$

Radial motion of photons

Integrating, we have:

$$ct = r + \frac{2GM}{c^2}\ln\left|\frac{rc^2}{2GM} - 1\right| + \text{constant outgoing photons,}$$

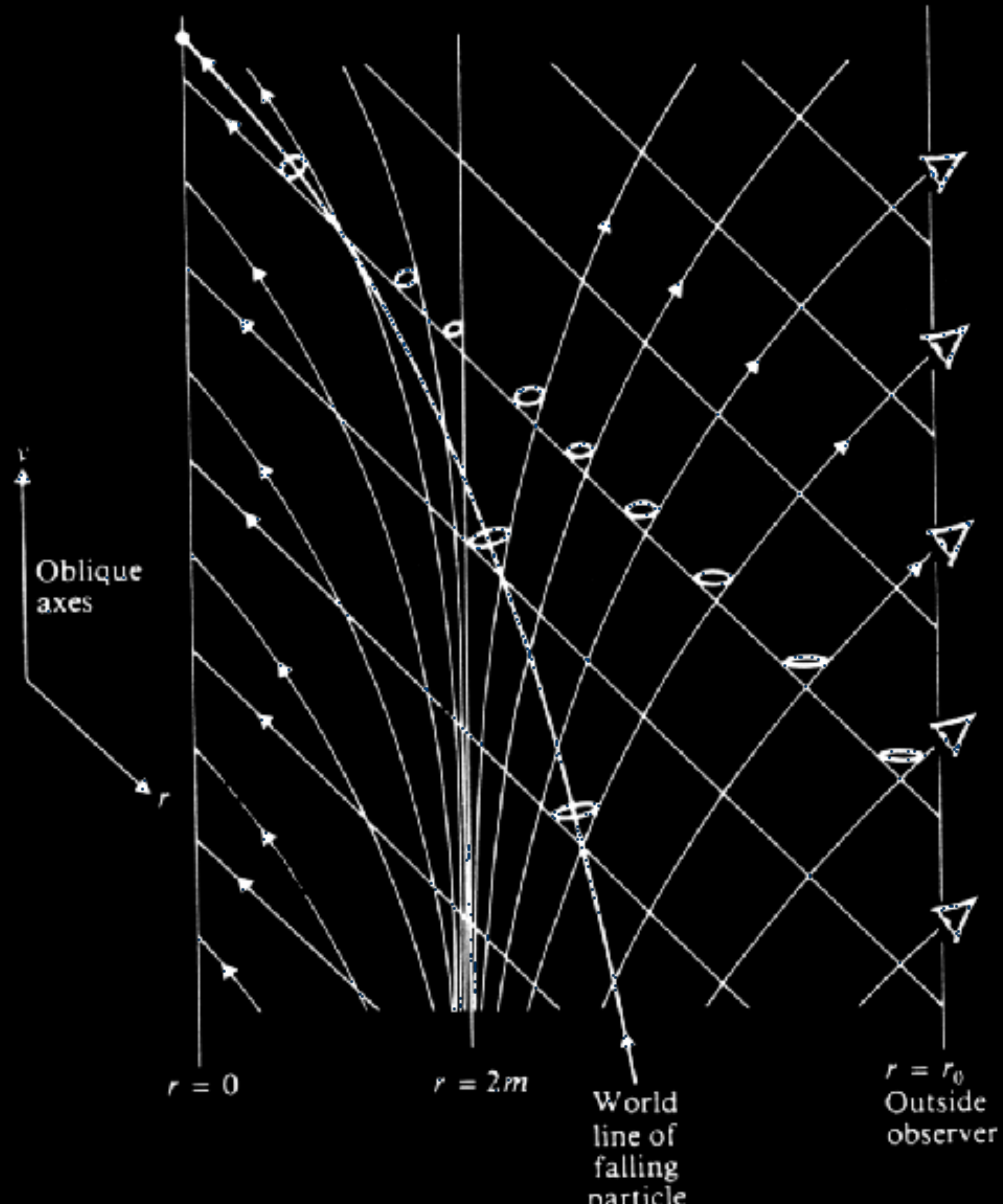
$$ct = -r - \frac{2GM}{c^2}\ln\left|\frac{rc^2}{2GM} - 1\right| + \text{constant incoming photons.}$$

Radial motion of photons

In a (ct, r)-diagram the photons have world-lines with slopes +/–1 as $r \rightarrow \infty$, indicating that space-time is asymptotically flat. As the events that generate the photons approach to $r = r_{\mathrm{Schw}}$, the slopes tend to +/– ∞. This means that the light cones become thinner and thinner for events close to the event horizon. At $r = r_{\mathrm{Schw}}$ the photons cannot escape and they move along the horizon.

An observer at infinity will never detect them.

Lightlike Geodesics Near the Event Horizon of a Black Hole



Circular motion of photons

In this case, fixing θ = constant due to the symmetry, we have that photons will move in a circle of r = constant and $ds^2 = 0$. Then,

$$\left(1-\frac{2GM}{rc^2}\right)c^2dt^2 - r^2d\phi^2 = 0.$$

This means that

$$\dot{\phi} = \frac{c}{r}\sqrt{\left(1-\frac{2GM}{rc^2}\right)} = \text{constant}.$$

Circular motion of photons

The circular velocity is:

$$v_{\mathrm{circ}} = \frac{r\dot{\phi}}{\sqrt{g_{00}}} = \frac{\Omega r}{(1 - 2GM/c^2 r)^{1/2}}.$$

Setting $v_{\mathrm{circ}} = c$ for photons and using $\Omega = (GM/r^3)^{1/2}$, we get that the only possible radius for a circular photon orbit is:

$$r_{\mathrm{ph}} = \frac{3GM}{c^2}.$$

Circular motion of photons

For a compact object of 1 M_{sun}, $r_{ph} \approx 4.5$ km, in comparison with the Schwarzschild radius of 3 km. Photons moving at this distance form the "photosphere" of the black hole.

The orbit, however, is unstable, as it can be seen from the effective potential:

$$V_{\mathrm{eff}} = \frac{L_{\mathrm{ph}}^2}{r^2}\left(1 - \frac{2r_{\mathrm{g}}}{r}\right).$$

Circular motion of photons

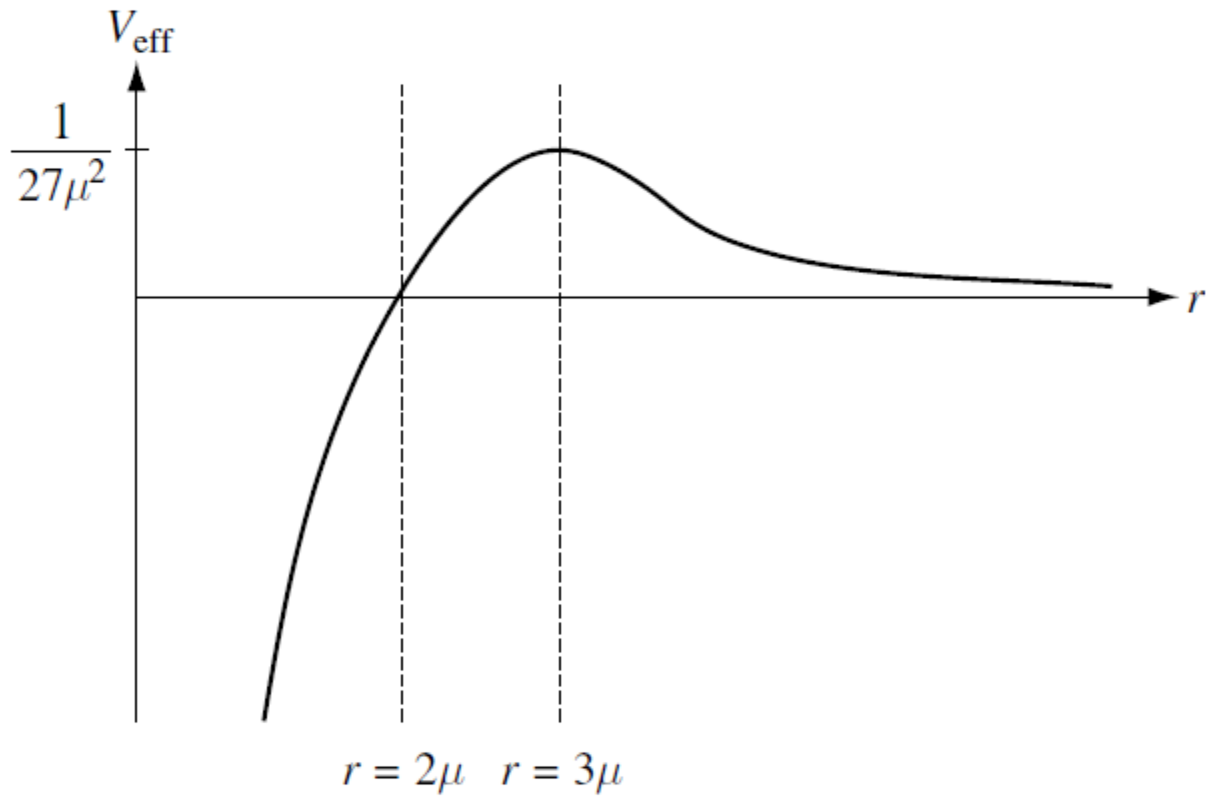
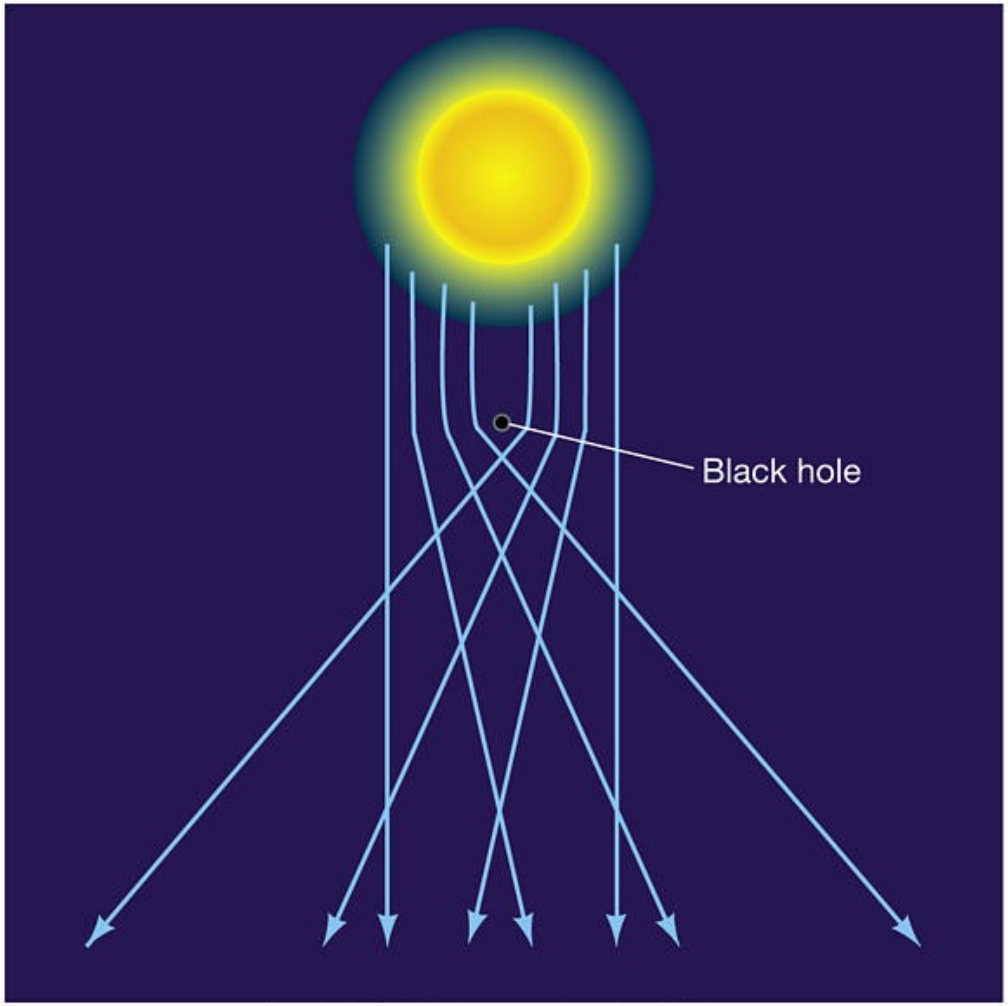
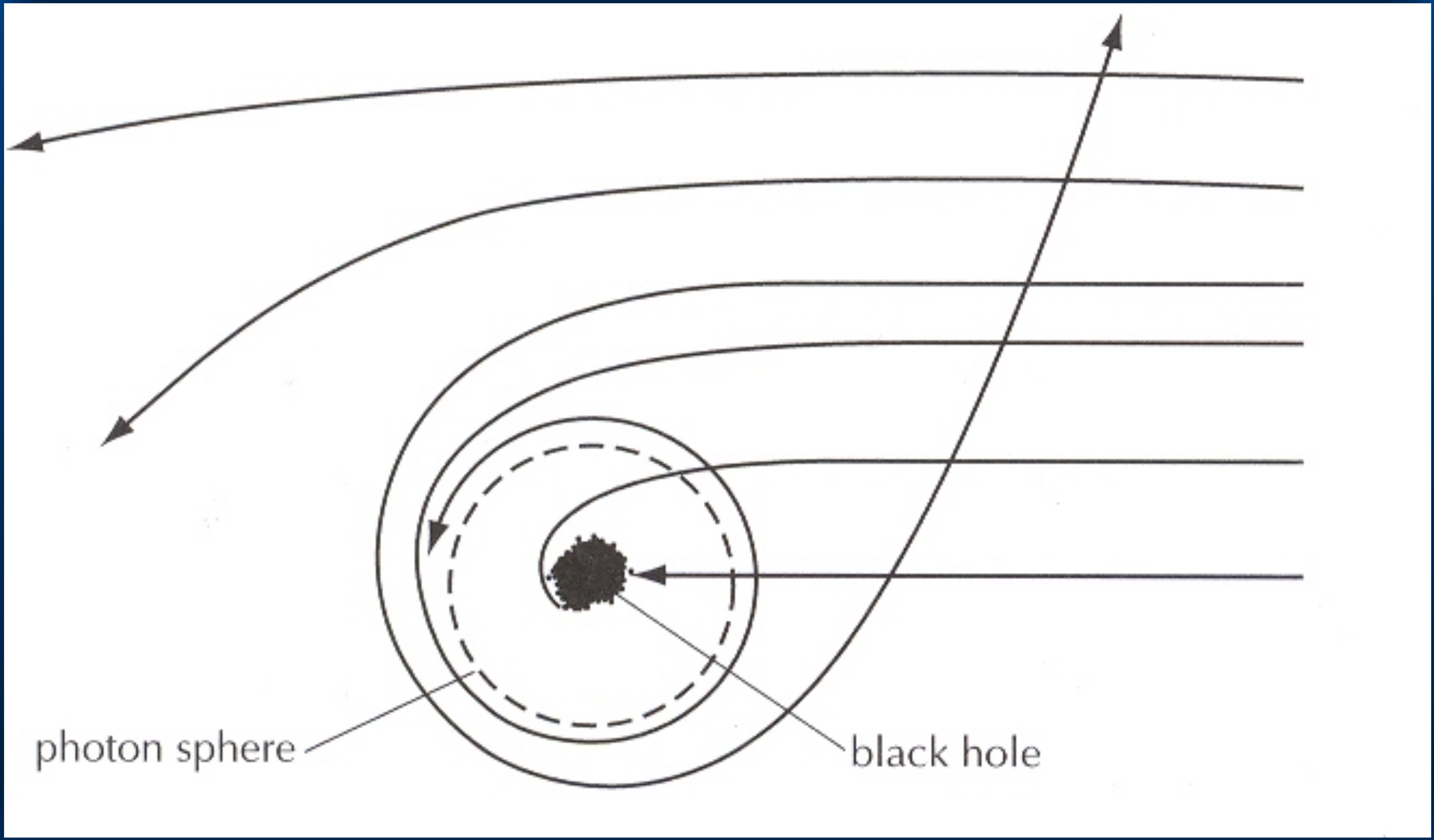


Figure The effective potential for photon orbits.

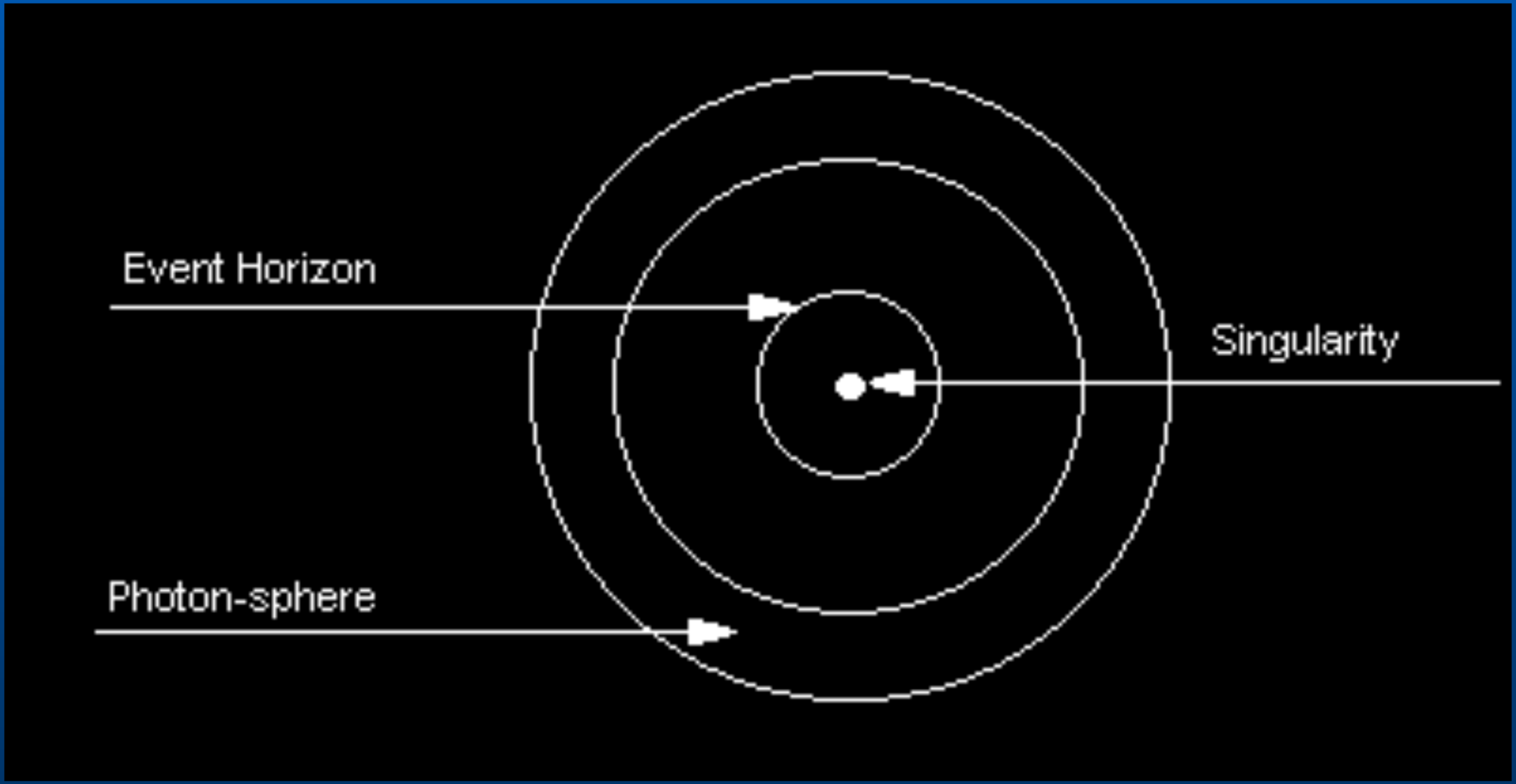
Motion of photons



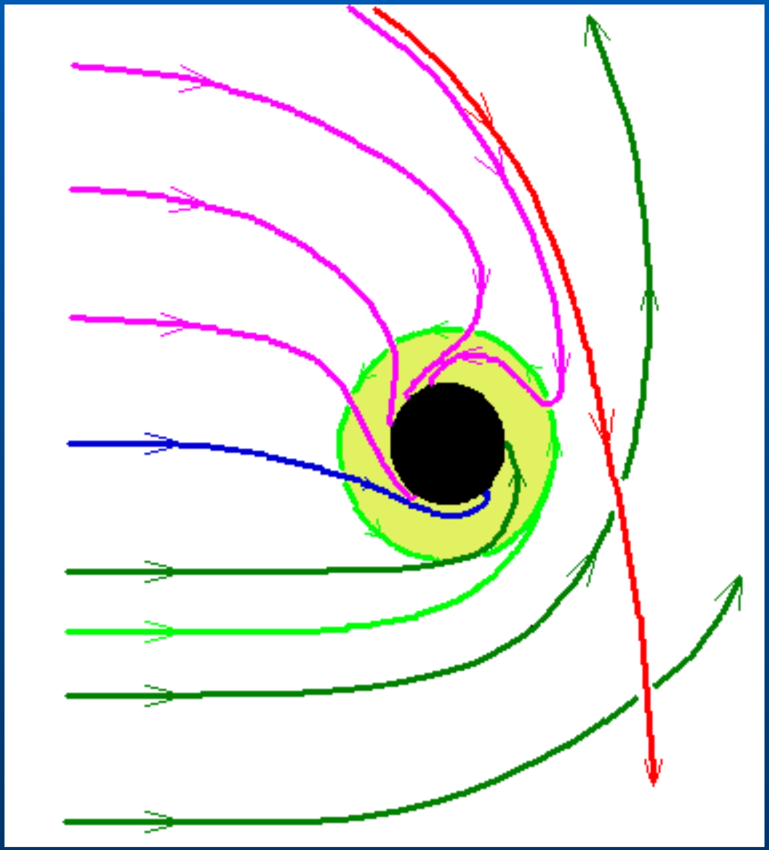
Circular motion of photons



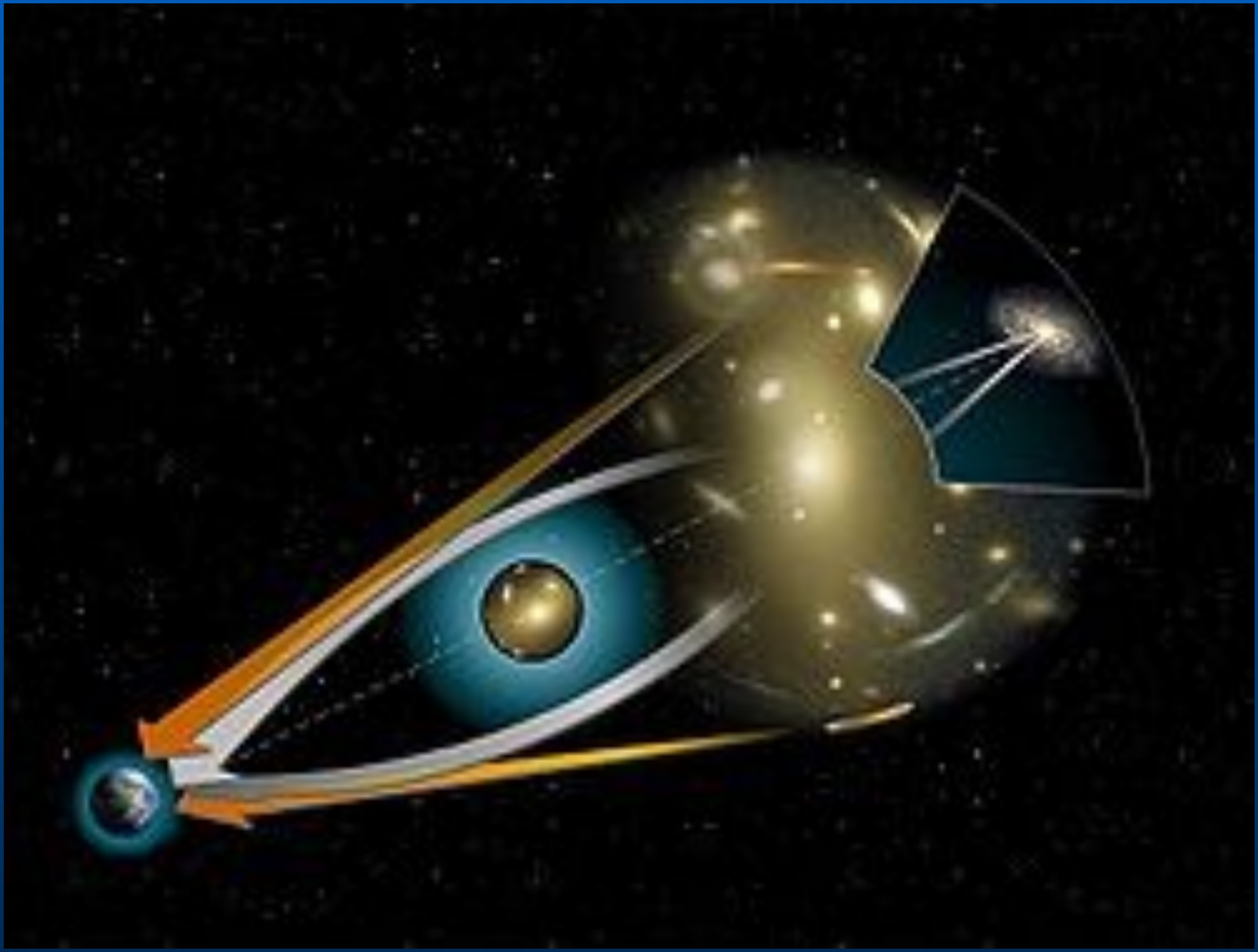
Circular motion of photons



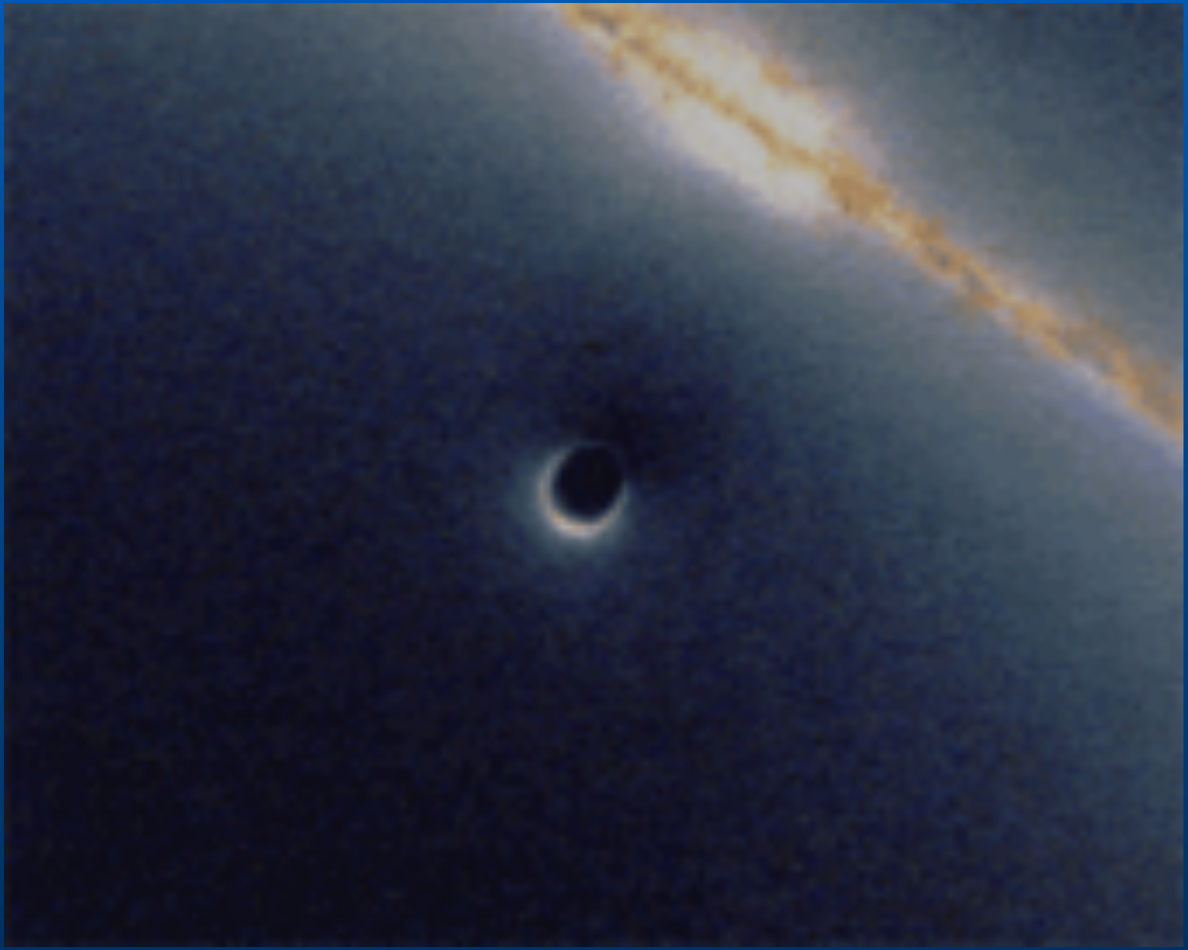
Circular motion of photons



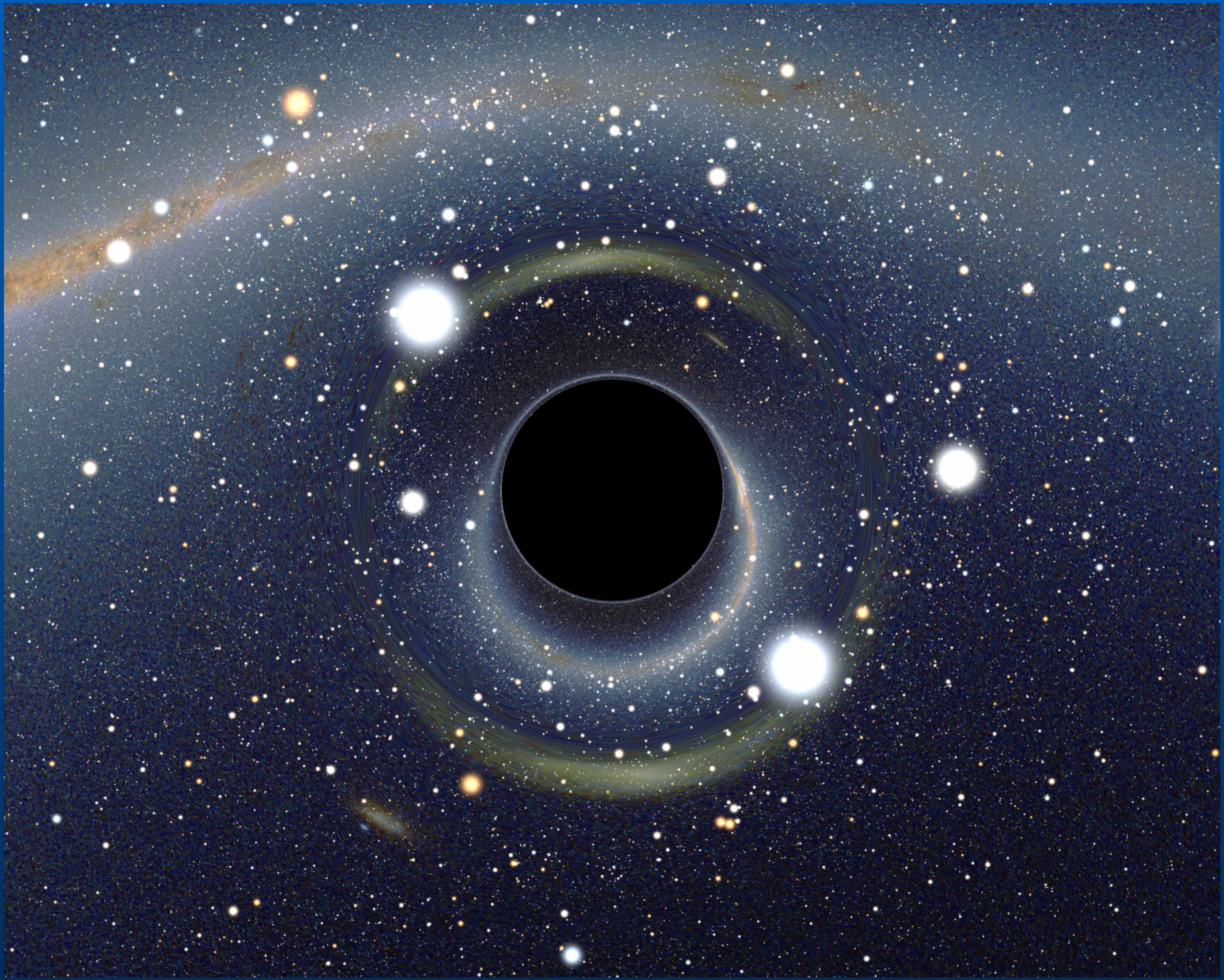
Gravitational lensing



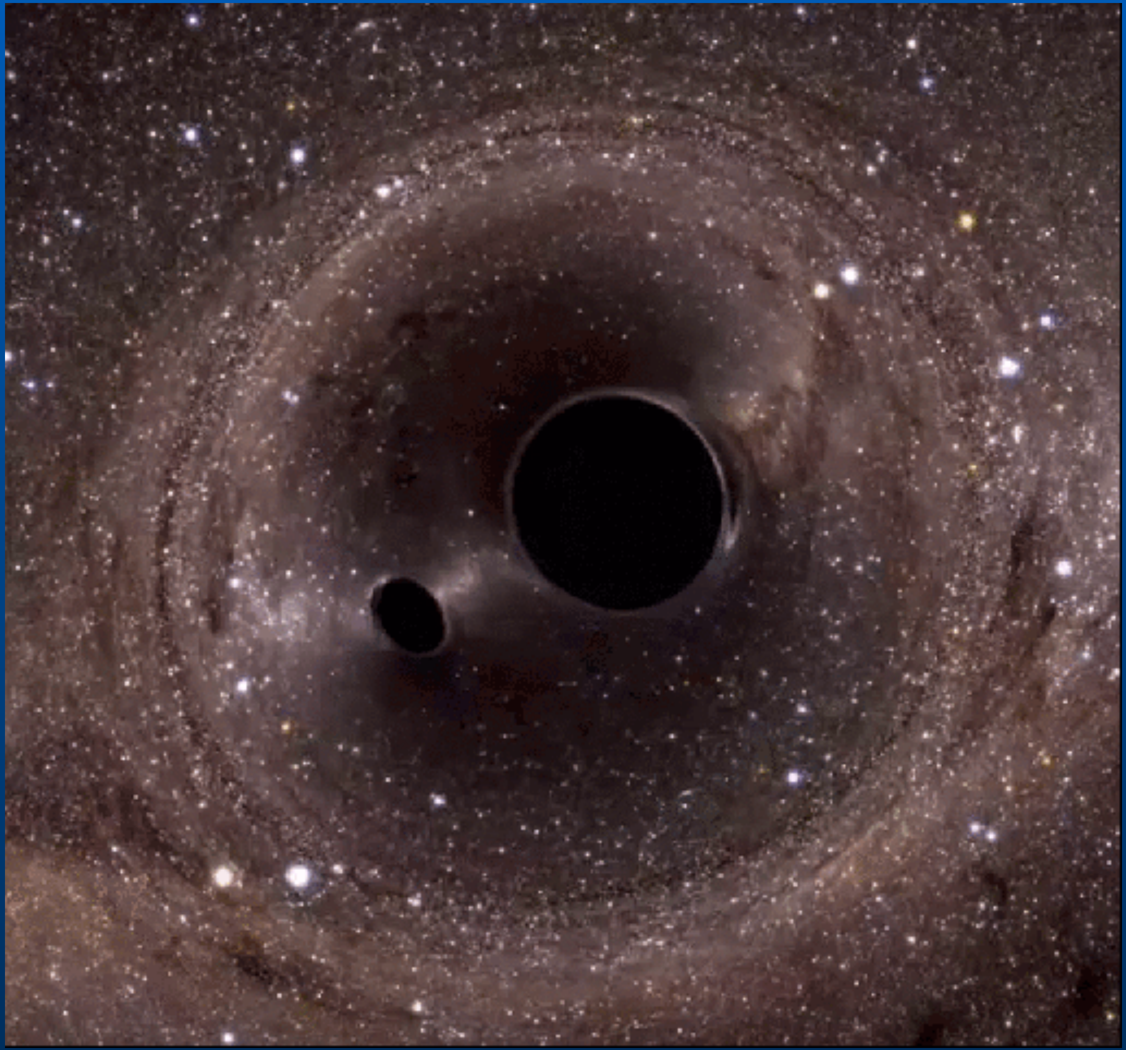
Gravitational lensing



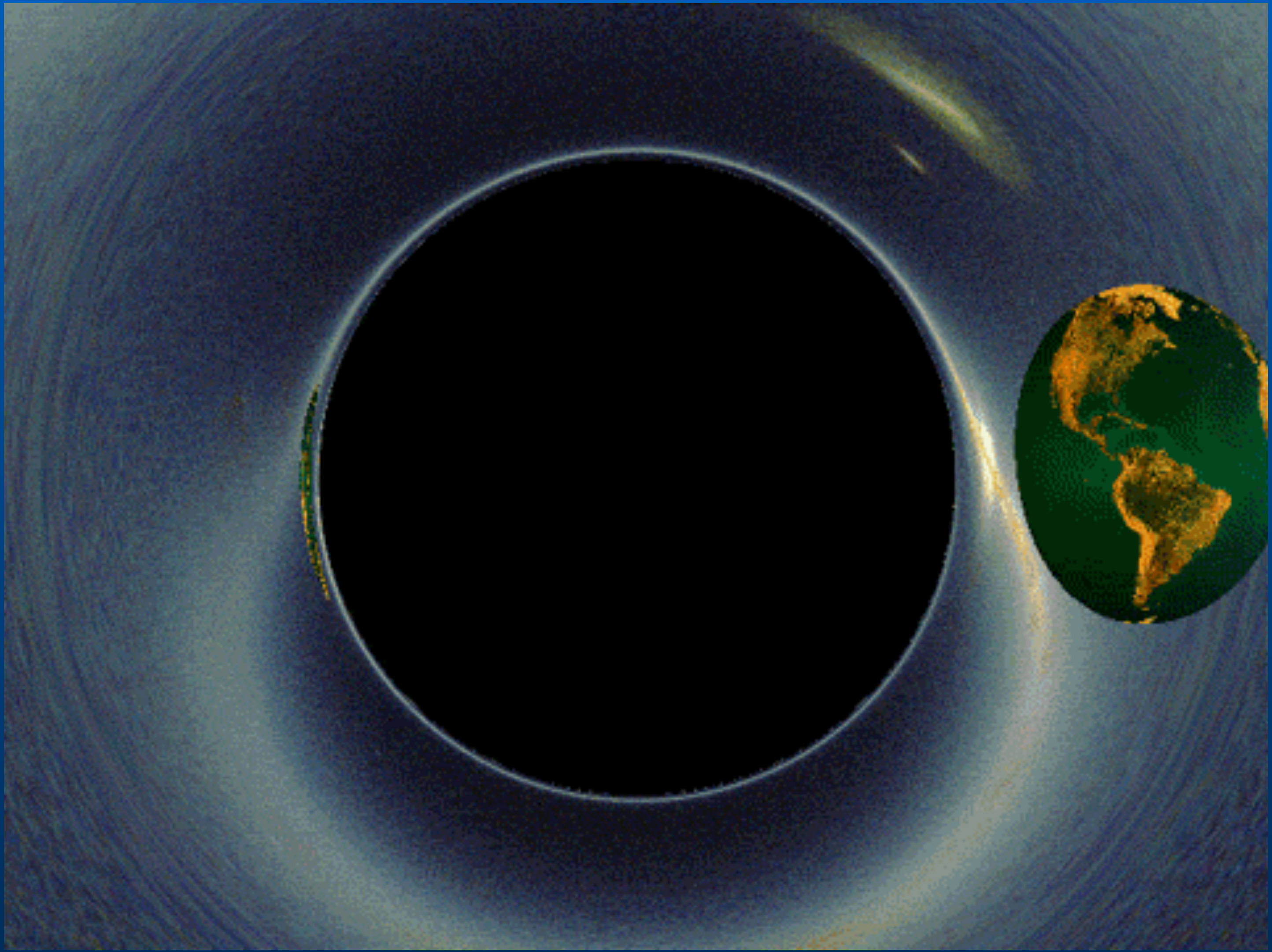
Gravitational lensing



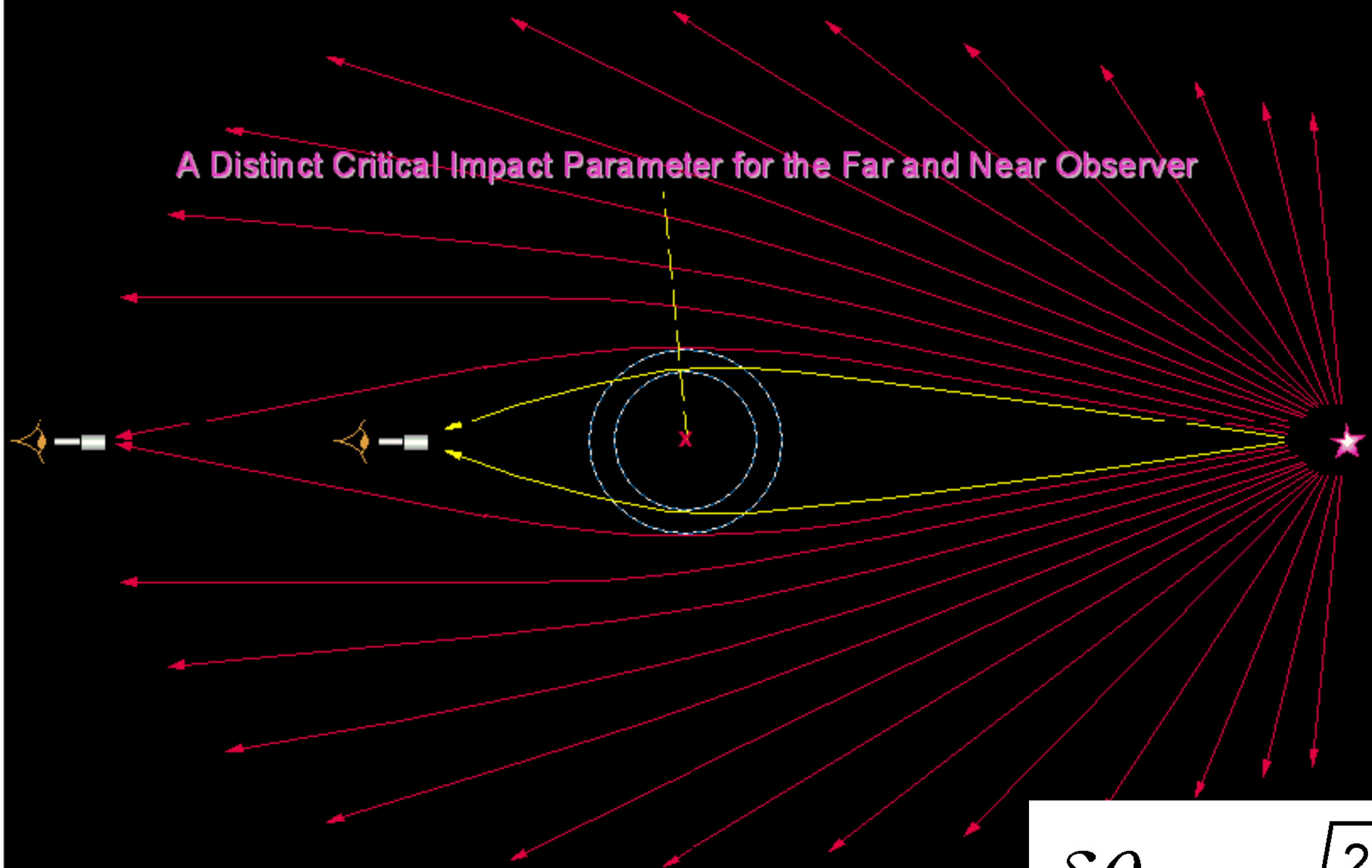
Gravitational lensing



Gravitational lensing

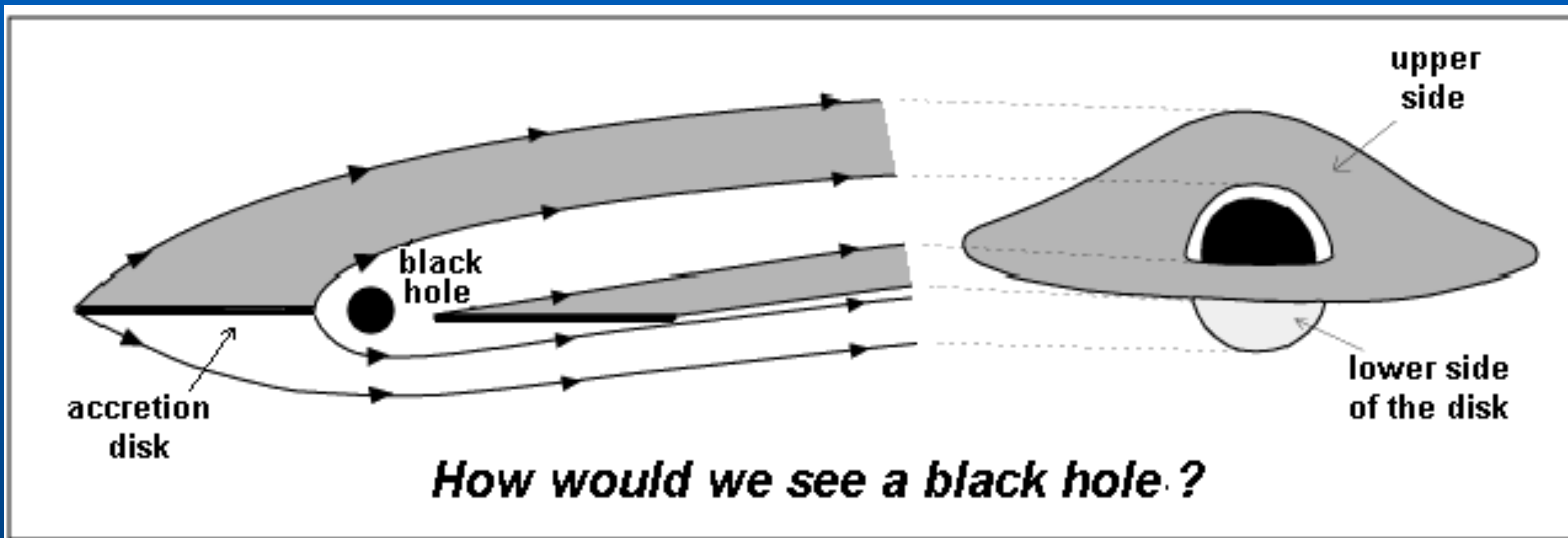


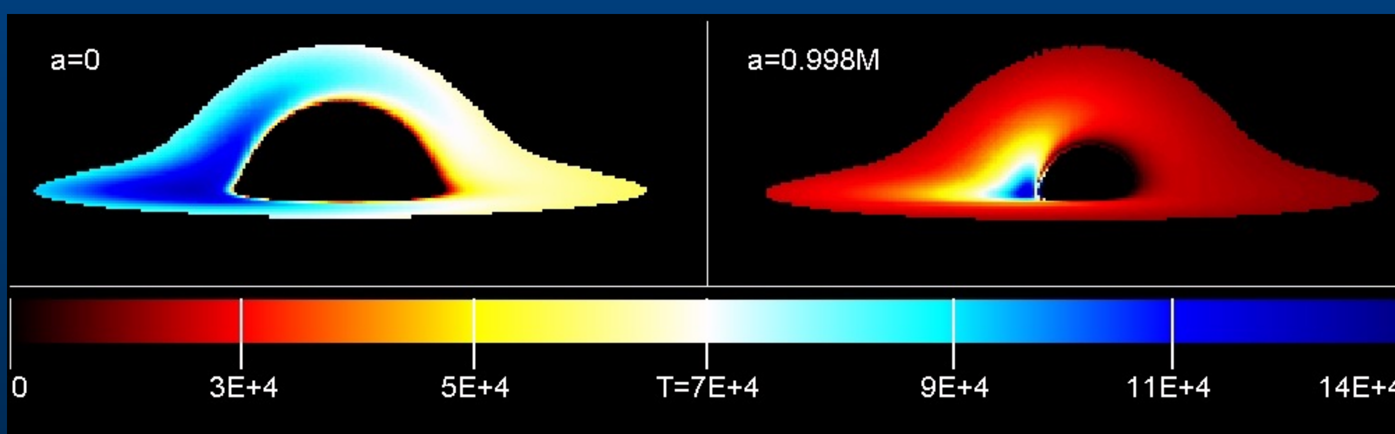
Principle of Reciprocity Demonstrated

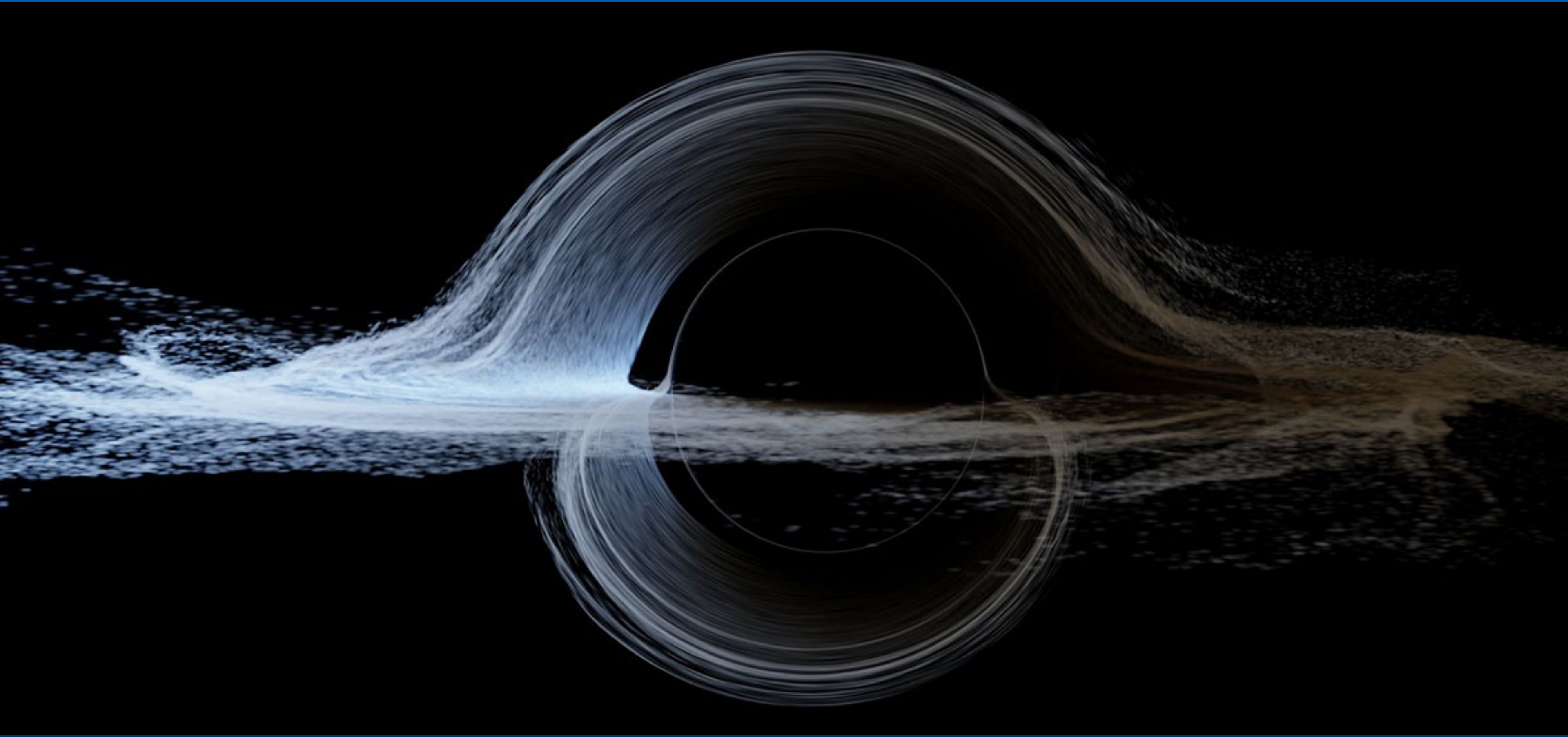


$$\delta\theta_1 = \sqrt{\frac{2GM}{D_L c^2}}$$

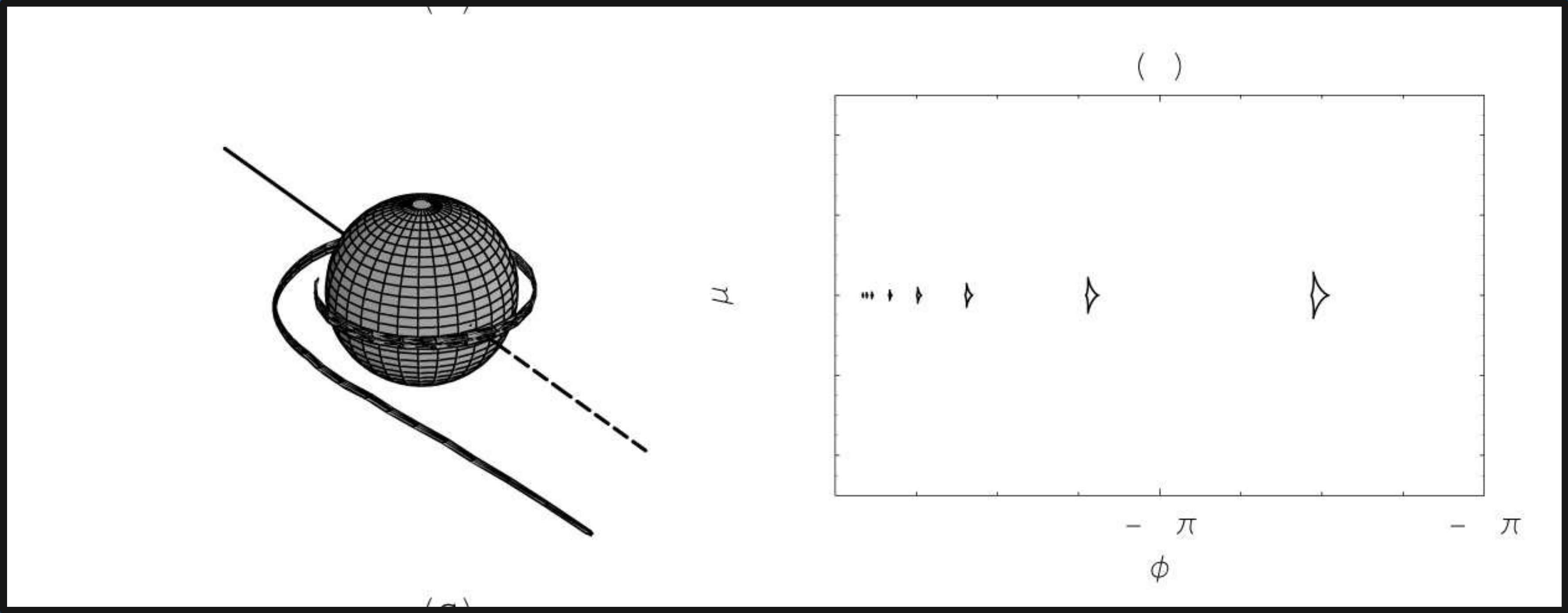
Gravitational lensing







Gravitational lensing



Gravitational capture

A particle coming from infinity is captured if its trajectory ends in the black hole. The angular momentum of a non-relativistic particle with velocity v_∞ at infinity is $L = mv_\infty b$, where b is an impact parameter. The condition $L/mcr_{\rm Schw} = 2$ defines $b_{\rm cr,non\text{-}rel} = 2r_{\rm Schw}(c/v_\infty)$. Then, the capture cross section is:

$$\sigma_{\rm non\text{-}rel} = \pi b_{\rm cr}^2 = 4\pi \frac{c^2 r_{\rm Schw}^2}{v_\infty^2}.$$

Gravitational capture

For an ultra-relativistic particle, $b_{\rm cr} = 3\sqrt{3} r_{\rm Schw}/2$, and then

$$\sigma_{\rm rel} = \pi b_{\rm cr}^2 = \frac{27}{4} \pi r_{\rm Schw}^2.$$

Orbits in Kerr spacetime

$$\theta = \pi/2$$

$$ds^2 = c^2\left(1-\frac{2\mu}{r}\right)dt^2 + \frac{4\mu ac}{r}\,dt\,d\phi - \frac{r^2}{\Delta}\,dr^2 - \left(r^2+a^2+\frac{2\mu a^2}{r}\right)d\phi^2$$

$$\dot{t} = \frac{1}{\Delta}\left[\left(r^2+a^2+\frac{2\mu a^2}{r}\right)k - \frac{2\mu a}{cr}h\right],$$

$$\dot{\phi} = \frac{1}{\Delta}\left[\frac{2\mu ac}{r}k + \left(1-\frac{2\mu}{r}\right)h\right].$$

$k \rightarrow E$, $h \rightarrow L$, when $a \rightarrow 0$

Orbits in Kerr spacetime

$$V_{\text{eff}}(r; h, k) = -\frac{\mu c^2}{r} + \frac{h^2 - a^2c^2(k^2-1)}{2r^2} - \frac{\mu(h-ack)^2}{r^3}.$$

$$k = \frac{1 - 2\mu u \mp a\sqrt{\mu u^3}}{(1 - 3\mu u \mp 2a\sqrt{\mu u^3})^{1/2}},$$

$$h = \mp \frac{c\sqrt{\mu}(1 + a^2u^2 \pm 2a\sqrt{\mu u^3})}{\sqrt{u}(1 - 3\mu u \mp 2a\sqrt{\mu u^3})^{1/2}}.$$

$$u = 1/r.$$

Orbits in Kerr spacetime

$$\frac{d^2V_{\rm eff}}{dr^2} = \frac{d^2V_{\rm eff}}{du^2}\left(\frac{du}{dr}\right)^2 + \frac{dV_{\rm eff}}{du}\frac{d^2u}{dr^2} = u^3\left(\frac{d^2V_{\rm eff}}{du^2} + 2\frac{dV_{\rm eff}}{du}\right) = 0,$$

$$r^2 - 6\mu r - 3a^2 \mp 8a\sqrt{\mu r} = 0,$$

$$\left(\frac{r_{\rm ms}}{GM/c^2}\right)^2 - 6\left(\frac{r_{\rm ms}}{GM/c^2}\right) \pm 8\left(\frac{r_{\rm ms}}{GM/c^2}\right)^{1/2} - 3 = 0.$$

Orbits in Kerr spacetime

For the "+" sign this is satisfied by $r_{\mathrm{ms}} = GM/c^2$, whereas for the "−" sign the solution is $r_{\mathrm{ms}} = 9GM/c^2$. The first case corresponds to a co-rotating particle and the second one to a counter-rotating particle. The energy of the co-rotating particle in the innermost orbit is $1/\sqrt{3}$ (units of mc^2). The binding energy of a particle in an orbit is the difference between the orbital energy and its energy at infinity. This means a binding energy of 42 % of the rest energy at infinity! For the counter-rotating particle, the binding energy is 3.8 %, smaller than for a Schwarzschild black hole.

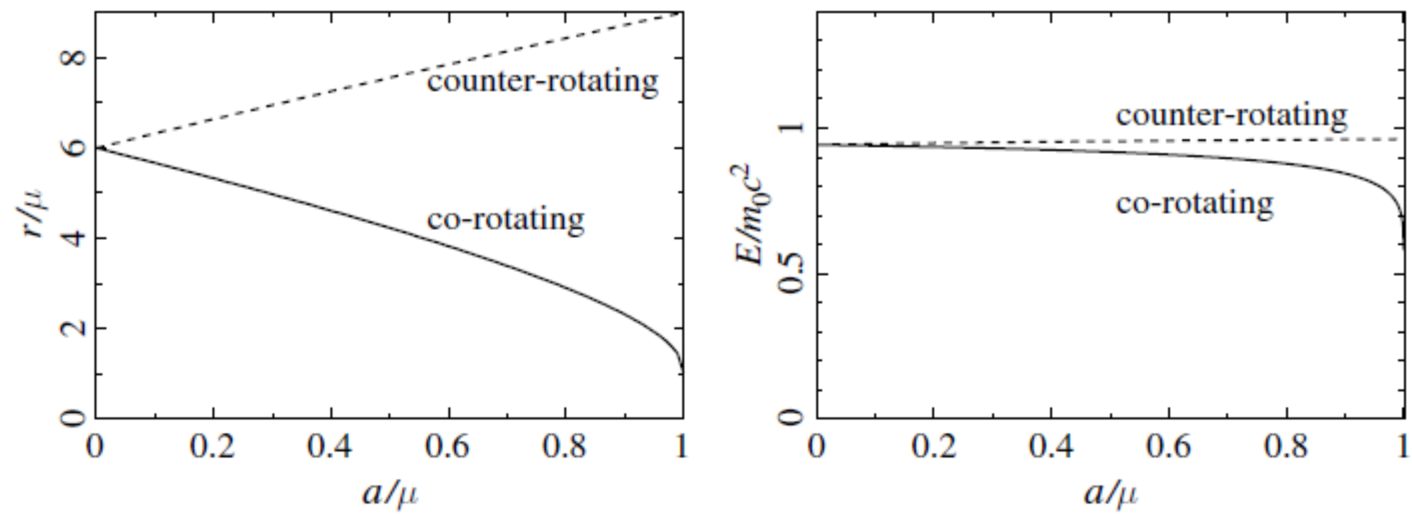


Figure 13.5 The scaled coordinate radii r/μ (left) and the constant $ks = E/(m_0c^2)$ (right) for the innermost stable co-rotating and counter-rotating circular orbits in the equatorial plane of the Kerr geometry, as functions of a/μ.

Orbits in Kerr spacetime

$$\frac{\dot{r}^2}{h^2} + V_{\text{eff}}(r; b) = \frac{1}{b^2},$$

$$b = h/(ck)$$

$$V_{\text{eff}}(r; b) = \frac{1}{r^2}\left[1 - \left(\frac{a}{b}\right)^2 - \frac{2\mu}{r}\left(1 - \frac{a}{b}\right)^2\right].$$

Orbits of photons in Kerr spacetime

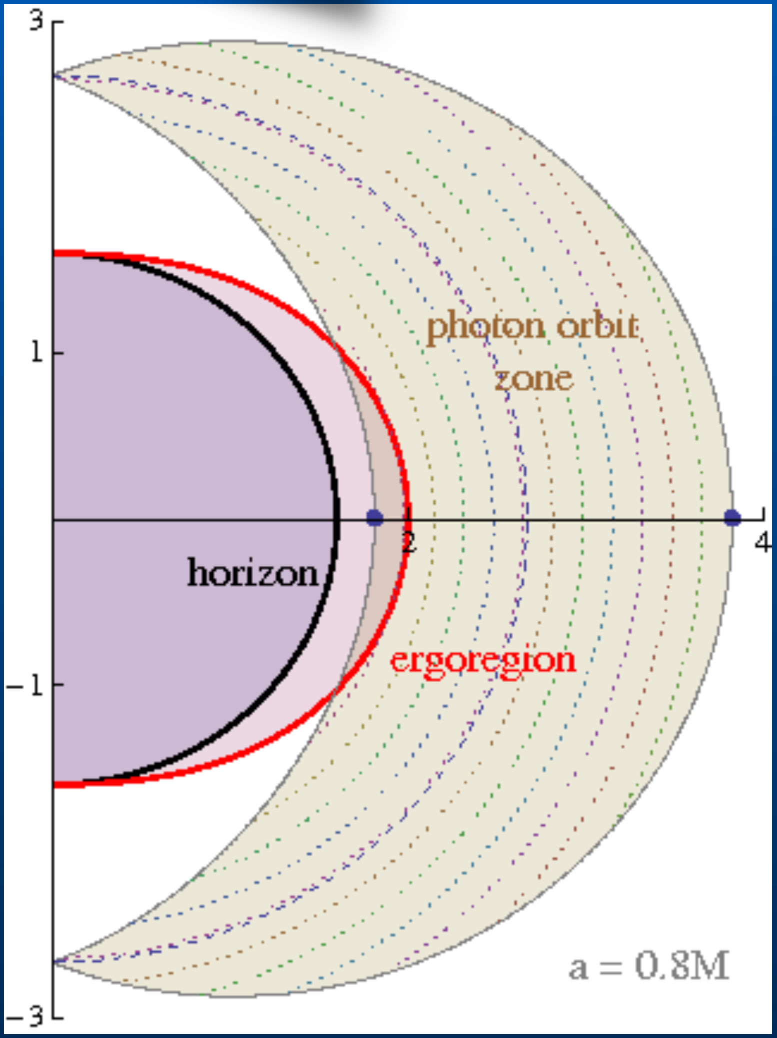
$$b = h/(ck)$$

$$V_{\text{eff}}(r_c; b) = \frac{1}{b^2} \quad \text{and} \quad \left.\frac{dV_{\text{eff}}}{dr}\right|_{r=r_c} = 0.$$

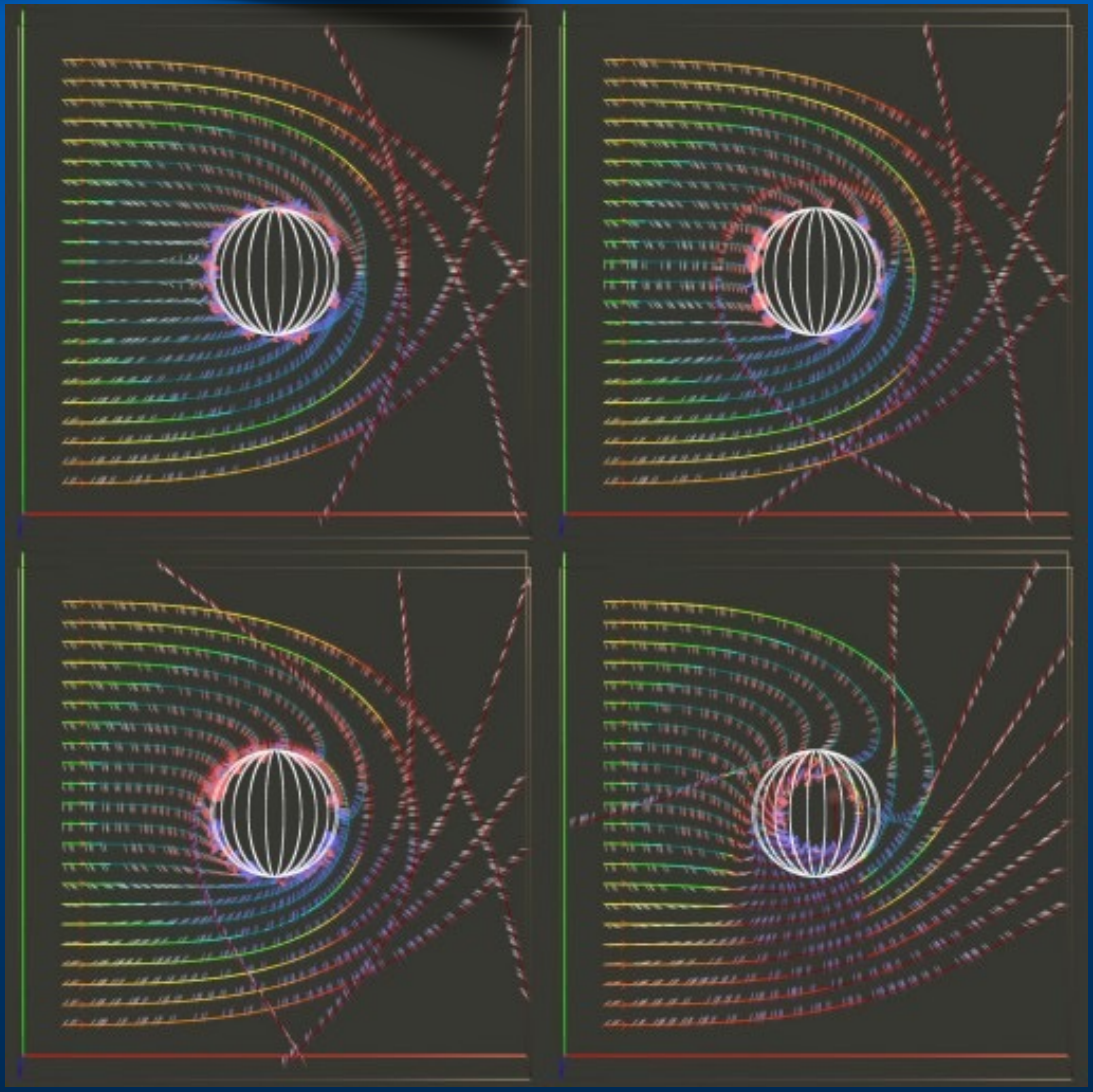
$$r_c = 2\mu\left\{1 + \cos\left[\frac{2}{3}\cos^{-1}\left(\pm\frac{a}{\mu}\right)\right]\right\},$$

$$b = 3\sqrt{\mu r_c} - a,$$

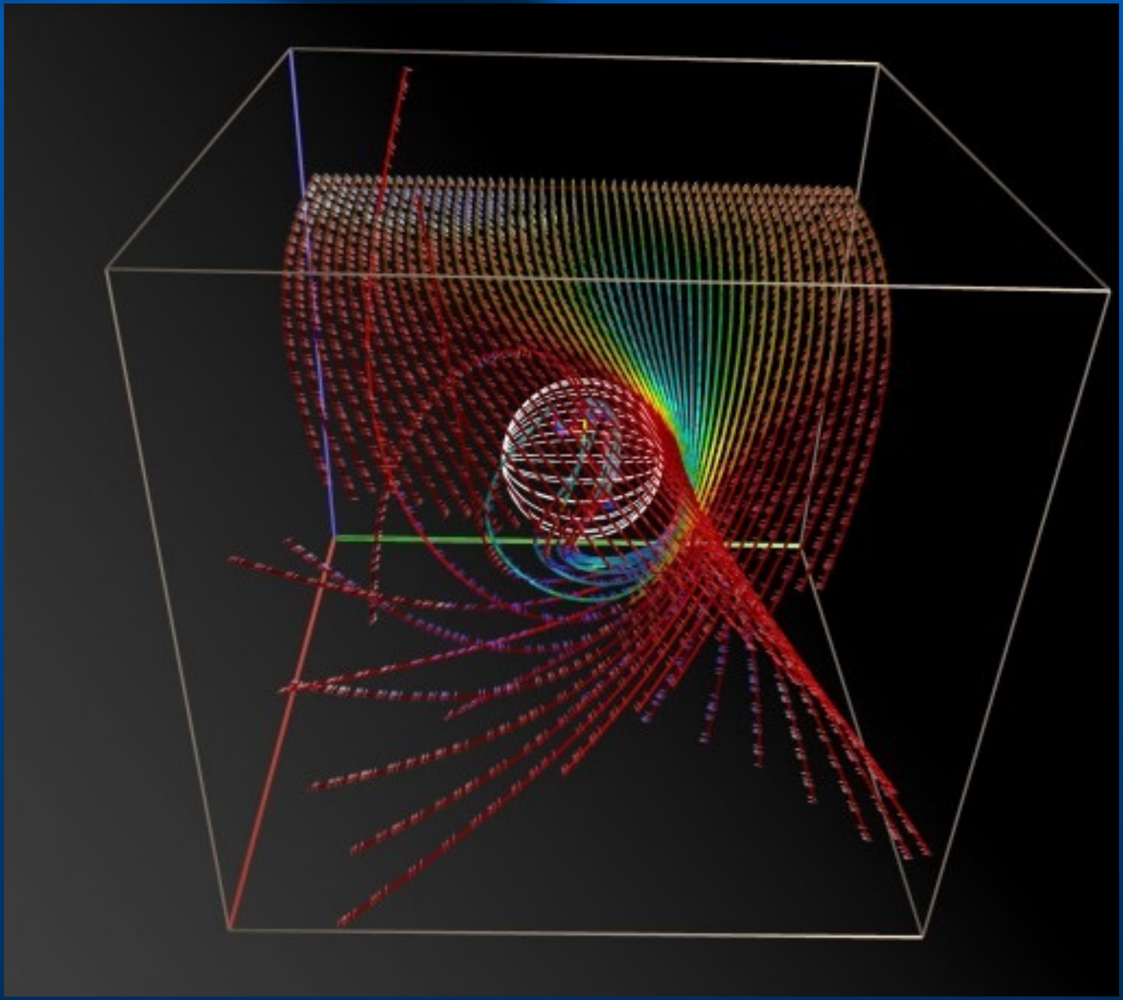
Orbits of photons in Kerr spacetime



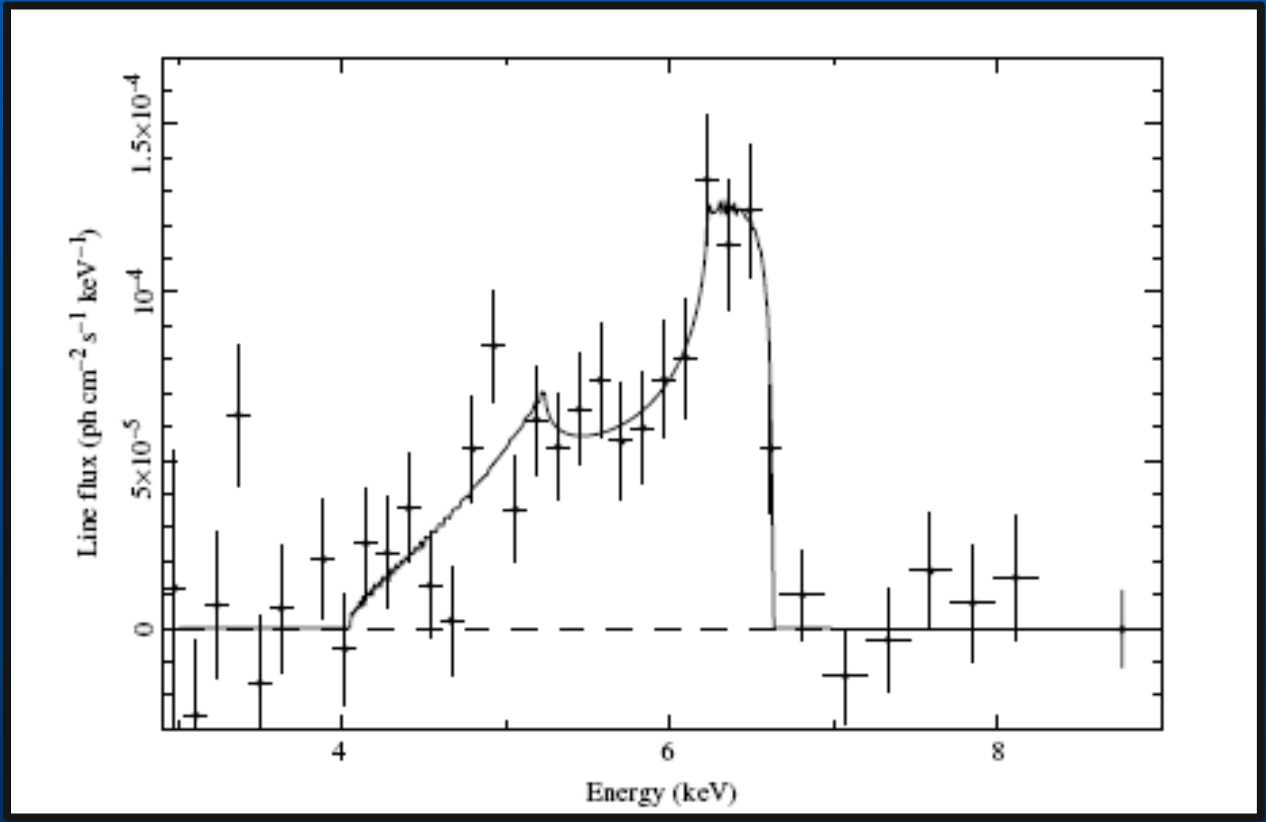
Orbits of photons in Kerr spacetime



Orbits of photons in Kerr spacetime



Effects of black hole spin



Pseudo-Newtonian potentials

The full effective general relativistic potential for particle orbits around a Kerr black hole is quite complex. Instead, pseudo-Newtonian potentials can be used. The first of such potentials, derived by Bohdam Paczyński and used by first time by Paczyński and Wiita (1980), for a non-rotating black hole with mass M, is:

$$\Phi = -\frac{GM}{r - 2r_g},$$

With this potential one can use Newtonian theory and obtain the same behavior of the Keplerian circular orbits of free particles as in the exact theory: orbits with $r < 9r_g$ are unstable, and orbits with $r < 6r_g$ are unbound. However, velocities of massive particles obtained with the potential are not accurate, since special relativistic effects are not included (Abramowicz et al. 1996).

Pseudo-Newtonian potentials

The velocity v_{p-N} calculated with the pseudo-Newtonian potential should be replaced by the corrected velocity $v_{\mathrm{corr}\ p-N}$ such that

$$v_{\mathrm{p-N}} = v_{\mathrm{p-N}}^{\mathrm{corr}} \gamma_{\mathrm{p-N}}^{\mathrm{corr}}, \qquad \gamma_{\mathrm{p-N}}^{\mathrm{corr}} = \frac{1}{\sqrt{1 - \left(\frac{v_{\mathrm{p-N}}^{\mathrm{corr}}}{c}\right)^2}}.$$

This re-scaling works amazingly well (see Abramowicz et al. 1996) compared with the actual velocities. The agreement with General Relativity is better than 5 %.

Black hole thermodynamics

The area of a Schwarzschild black hole is

$$A_{\text{Schw}} = 4\pi r_{\text{Schw}}^2 = \frac{16\pi G^2 M^2}{c^4}.$$

When a black hole absorbs a mass δM, its mass increases to $M + \delta M$, and hence, the area increases as well. Since the horizon can be crossed in just one direction the area of a black hole can only increase. This suggests an analogy with entropy.

Black hole thermodynamics

$$\delta S = \frac{\delta Q}{T_{\rm BH}} = \frac{\delta M c^2}{T_{\rm BH}}.$$

Particles trapped in the black hole will have a wavelength:

$$E_{ph} = h\nu = hc/\lambda = kT \qquad \lambda = \frac{\hbar c}{kT} \propto r_{\rm Schw},$$

where k is the Boltzmann constant. Then,

$$\xi \frac{\hbar c}{kT} = \frac{2GM}{c^2},$$

where ξ is a numerical constant. Then,

$$T_{\rm BH} = \xi \frac{\hbar c^3}{2GkM}, \quad \text{and} \quad S = \frac{c^6}{32\pi G^2 M} \int \frac{dA_{\rm Schw}}{T_{\rm BH}} = \frac{c^3 k}{16\pi\hbar G\xi} A_{\rm Schw} + \text{ constant}.$$

Black hole thermodynamics

A quantum mechanical calculation of the horizon temperature in the Schwarzschild case leads to $\xi = (4\pi)^{-1}$. So,

$$T_{\mathrm{BH}} = \frac{\hbar c^3}{8GMk} \cong 10^{-7}\,\mathrm{K}\left(\frac{M_\odot}{M}\right).$$

And we can write the entropy of the black hole as:

$$S = \frac{kc^3}{4\pi\hbar G} A_{\mathrm{Schw}} + \text{constant} \sim 10^{77}\left(\frac{M}{M_\odot}\right)^2 k\,\mathrm{J\,K^{-1}}.$$

Black hole thermodynamics

The formation of a black hole implies a huge increase of entropy. Just to compare, a star has an entropy ~20 orders of magnitude lower than the corresponding black hole. This tremendous increase of entropy is related to the loss of all the structure of the original system (e.g. a star) once the black hole is formed.

TABLE 1
CURRENT ENTROPY OF THE OBSERVABLE UNIVERSE (SCHEME 1 ENTROPY BUDGET)

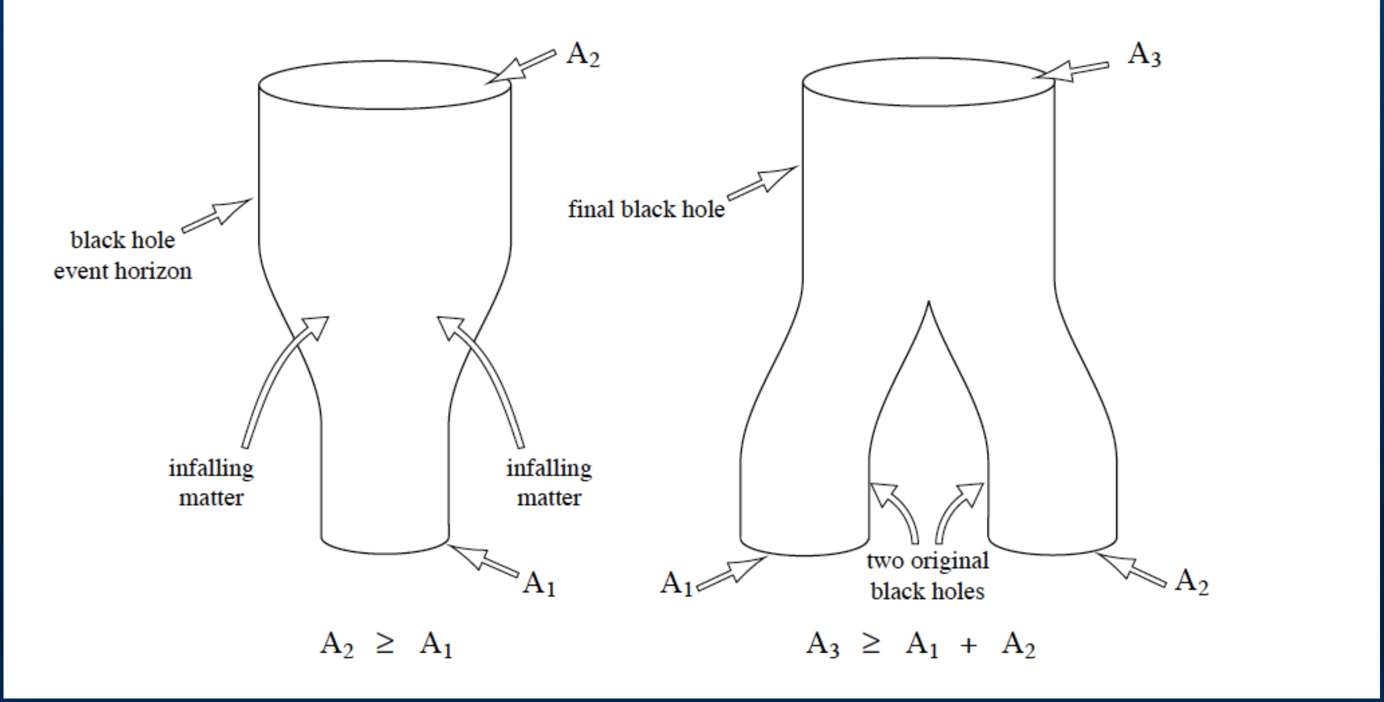
Component	Entropy Density s [$k\,m^{-3}$]	Entropy S [k]	Entropy S [k] (Previous Work)
SMBHs	$8.4^{+8.2}_{-4.7} \times 10^{23}$	$3.1^{+3.0}_{-1.7} \times 10^{104}$	10^{101}[1], 10^{102}[2], 10^{103}[3]
Stellar BHs ($2.5-15\ M_\odot$)	$1.6 \times 10^{17^{+0.6}_{-1.2}}$	$5.9 \times 10^{97^{+0.6}_{-1.2}}$	10^{97}[2], 10^{98}[4]
Photons	$1.478 \pm 0.003 \times 10^{9}$	$5.40 \pm 0.15 \times 10^{89}$	10^{88}[1,2,4], 10^{89}[5]
Relic Neutrinos	$1.411 \pm 0.014 \times 10^{9}$	$5.16 \pm 0.15 \times 10^{89}$	10^{88}[2],10^{89}[5]
WIMP Dark Matter	$5 \times 10^{7\pm1}$	$2 \times 10^{88\pm1}$	–
Relic Gravitons	$1.7 \times 10^{7^{+0.2}_{-2.5}}$	$6.2 \times 10^{87^{+0.2}_{-2.5}}$	10^{86}[2,3]
ISM and IGM	20 ± 15	$7.1 \pm 5.6 \times 10^{81}$	–
Stars	0.26 ± 0.12	$9.5 \pm 4.5 \times 10^{80}$	10^{79}[2]
Total	$\mathbf{8.4^{+8.2}_{-4.7} \times 10^{23}}$	$\mathbf{3.1^{+3.0}_{-1.7} \times 10^{104}}$	**10^{101}[1], 10^{102}[2], 10^{103}[3]**
Tentative Components:			
Massive Halo BHs ($10^5\ M_\odot$)	10^{25}	10^{106}	10^{106}[6]
Stellar BHs ($42-140\ M_\odot$)	$8.5 \times 10^{18^{+0.8}_{-1.6}}$	$3.1 \times 10^{99^{+0.8}_{-1.6}}$	–

NOTE. — Our budget is consistent with previous estimates from the literature with the exception that SMBHs, which dominate the budget, contain at least an order of magnitude more entropy as previously estimated, due to the contributions of black holes 100 times larger than those considered in previous budgets. Uncertainty in the volume of the observable universe (see Appendix) has been included in the quoted uncertainties. Massive halo black holes at $10^5\ M_\odot$ and stellar black holes in the range $42-140\ M_\odot$ are included tentatively since their existence is speculative. They are not counted in the budget totals. Previous work: [1] Penrose (2004), [2] Frampton et al. (2009), [3] Frampton & Kephart (2008), [4] Frautschi (1982), [5] Kolb & Turner (1981), [6] Frampton (2009b).

Black holes dominate the observable entropy of the universe

Black hole thermodynamics: The four laws

- First law (energy conservation): $dM = T_{\rm BH} dS + \Omega_+ dJ + \Phi dQ$. Here, Ω is the angular velocity and Φ the electrostatic potential.
- Second law (entropy never decreases): In all physical processes involving black holes the total surface area of all the participating black holes can never decrease.



Black hole thermodynamics: The four laws

- Third law (Nernst's law): The temperature (surface gravity) of a black black hole cannot be zero. Since $T_{\mathrm{BH}} = 0$ with $A \neq 0$ for extremal charged and extremal Kerr black holes, these are thought to be limit cases that cannot be reached in Nature.

- Zeroth law (thermal equilibrium): The surface gravity (temperature) is constant over the event horizon of a stationary axially symmetric black hole.

Quantum effects in black holes

If a temperature can be associated with black holes, then they should radiate as any other body. The luminosity of a Schwarzschild black hole is:

$$L_{\rm BH} = 4\pi r_{\rm Schw}^2 \sigma T_{\rm BH}^4 \sim \frac{16\pi\sigma\hbar^4 c^6}{(8\pi)^4 G^2 M^2 k^4}.$$

Here, σ is the Stephan-Boltzmann constant. This expression can be written as:

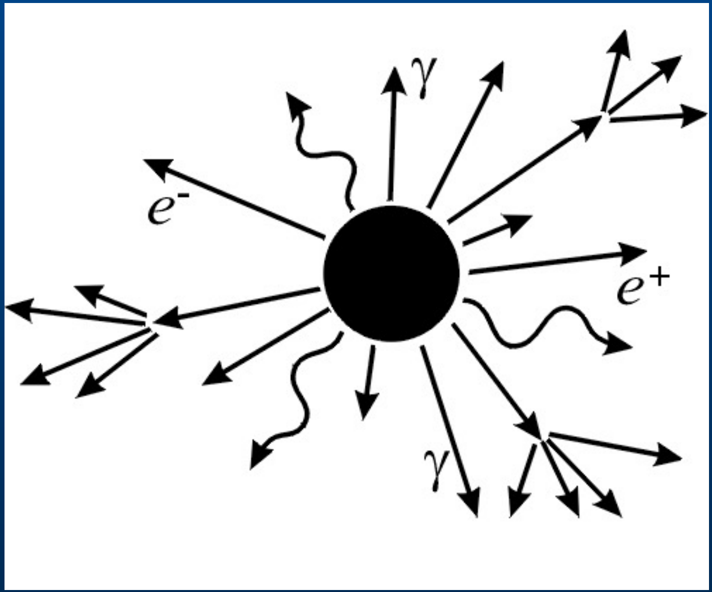
$$L_{\rm BH} = 10^{-17} \left(\frac{M_\odot}{M}\right)^2 \quad {\rm erg\ s^{-1}}.$$

Quantum effects in black holes

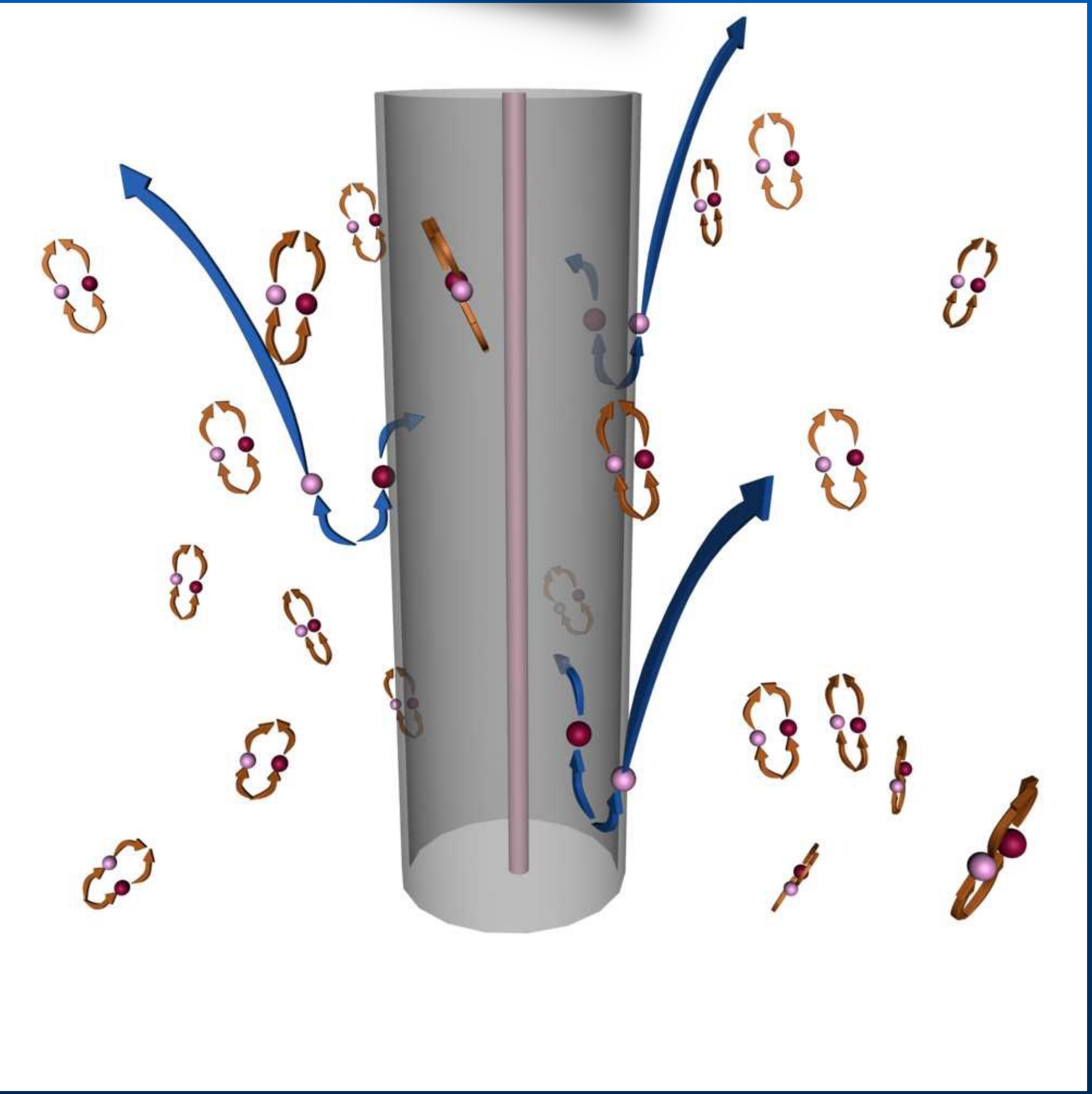
The lifetime of a black hole is:

$$\tau \cong \frac{M}{dM/dt} \sim 2.5 \times 10^{63} \left(\frac{M}{M_{\odot}}\right)^3 \quad \text{years.}$$

Notice that the black hole heats up as it radiates!. This occurs because when the hole radiates, its mass decreases and then according to Eq. (235) the temperature must rise.



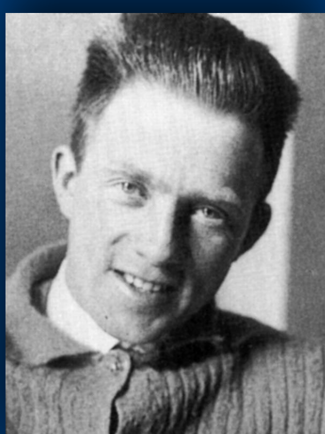
Quantum effects in black holes

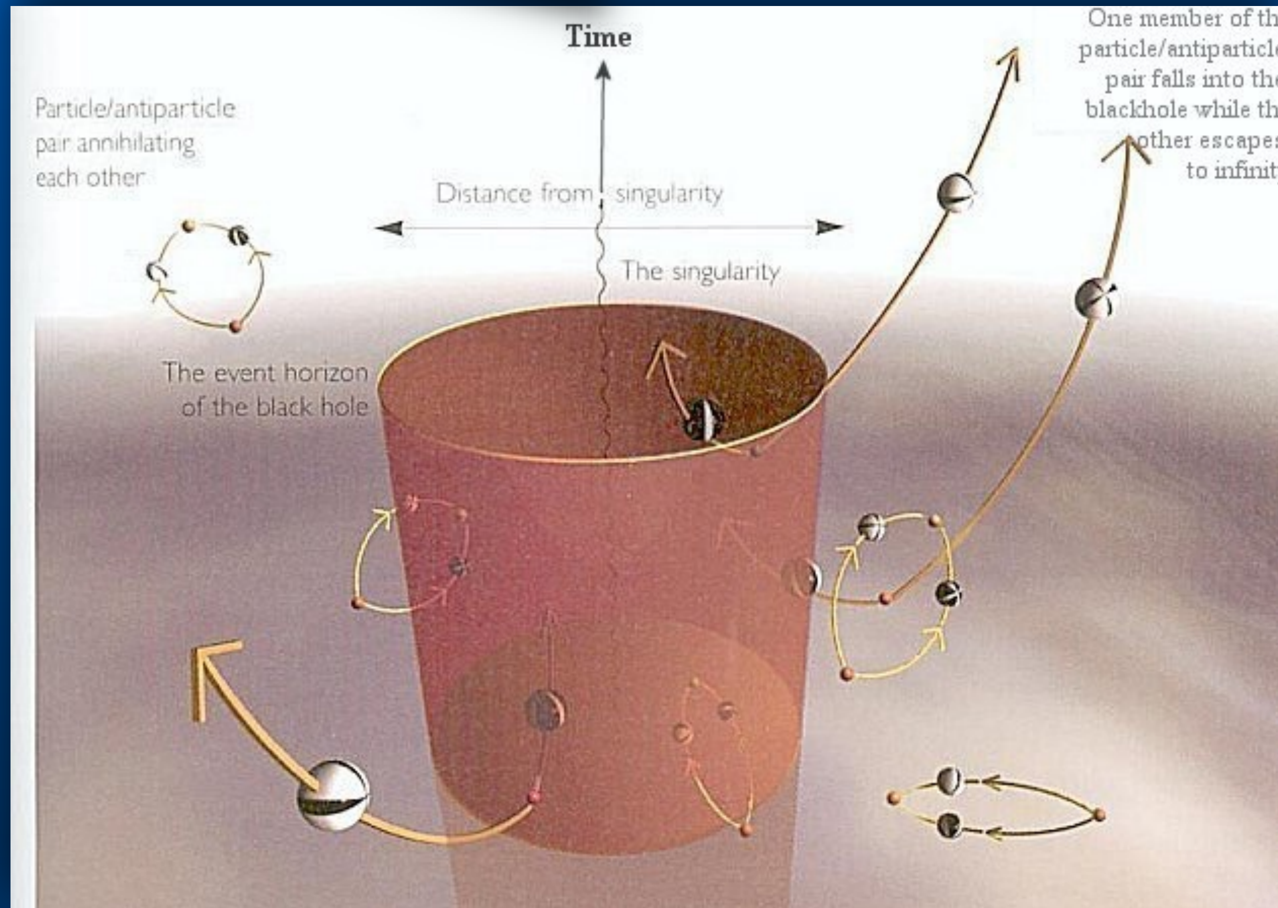


Quantum effects in black holes

$$\Delta p\,\Delta x \geq \frac{1}{2}\hbar$$

$$\Delta_\psi E \frac{\Delta_\psi B}{\left|\frac{\mathrm{d}\langle \hat{B}\rangle}{\mathrm{d}t}\right|} \geq \frac{\hbar}{2}$$





This picture is a misleading interpretation of the actual situation. The radiation is electromagnetic and not in the form of pairs, at least for astrophysical black holes.

Novikov & Frolov

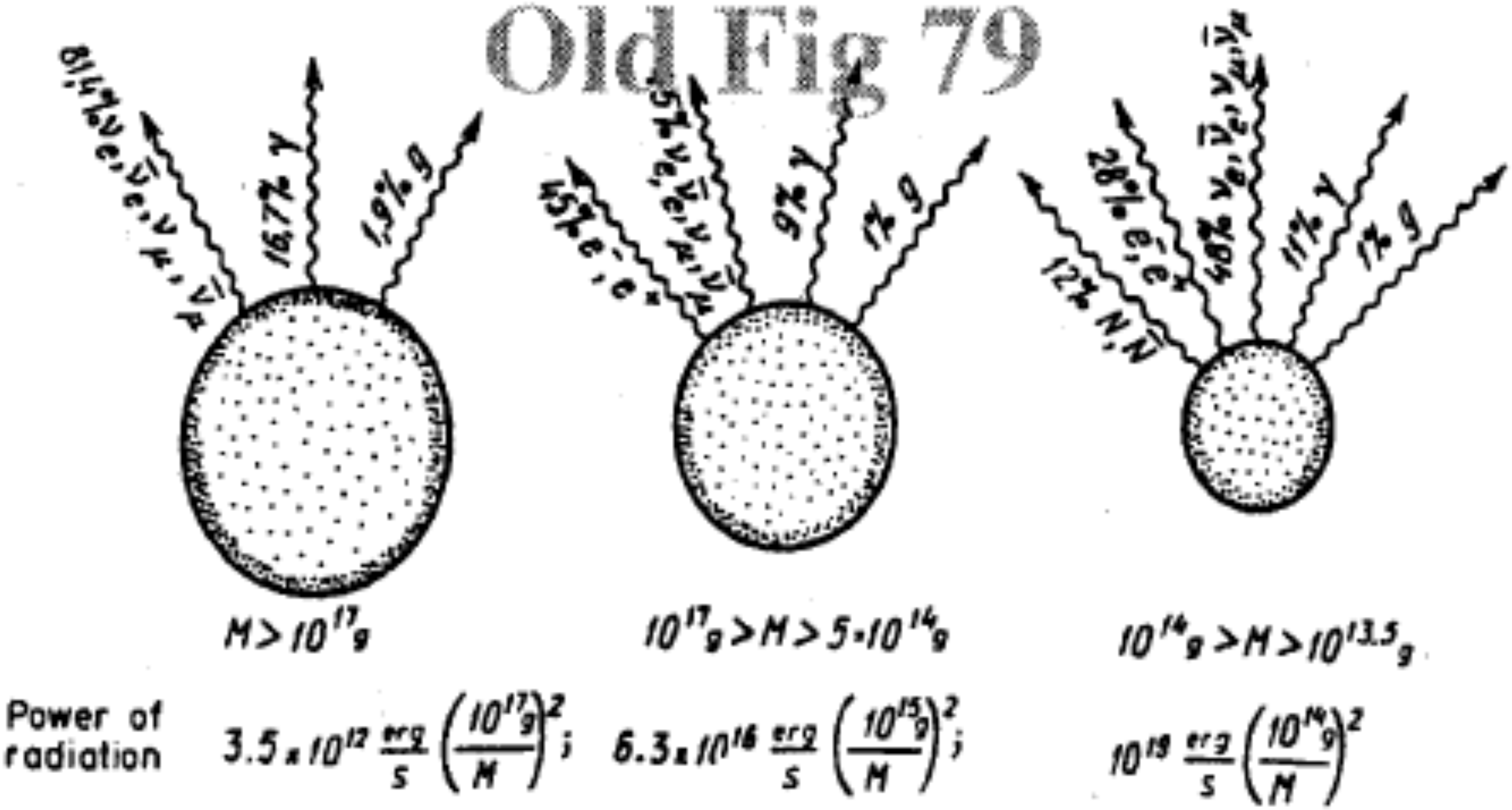


Figure 10.7: Quantum decay of a non-rotating black hole. The fractions of gravitons (g), photons (γ), neutrinos (ν) and other elementary particles are given in percent of the total number of particles emitted by black holes of different masses.

Old Fig 78

Novikov & Frolov

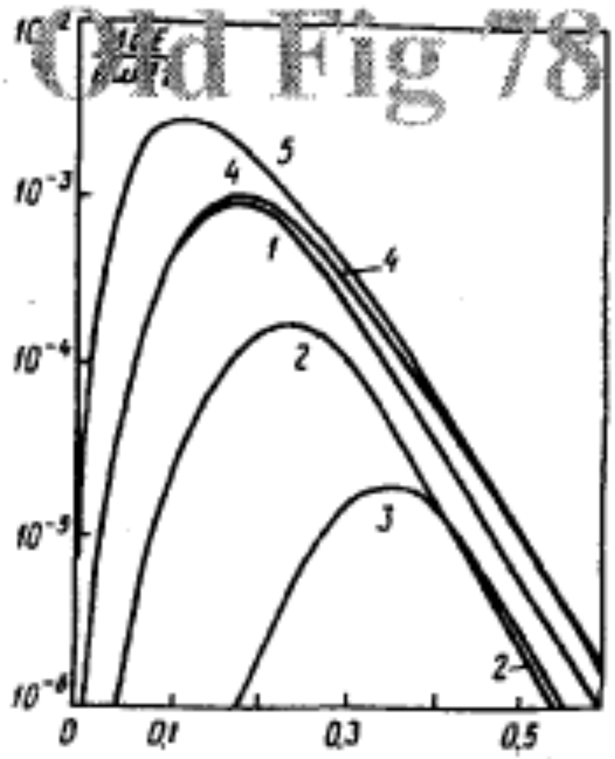
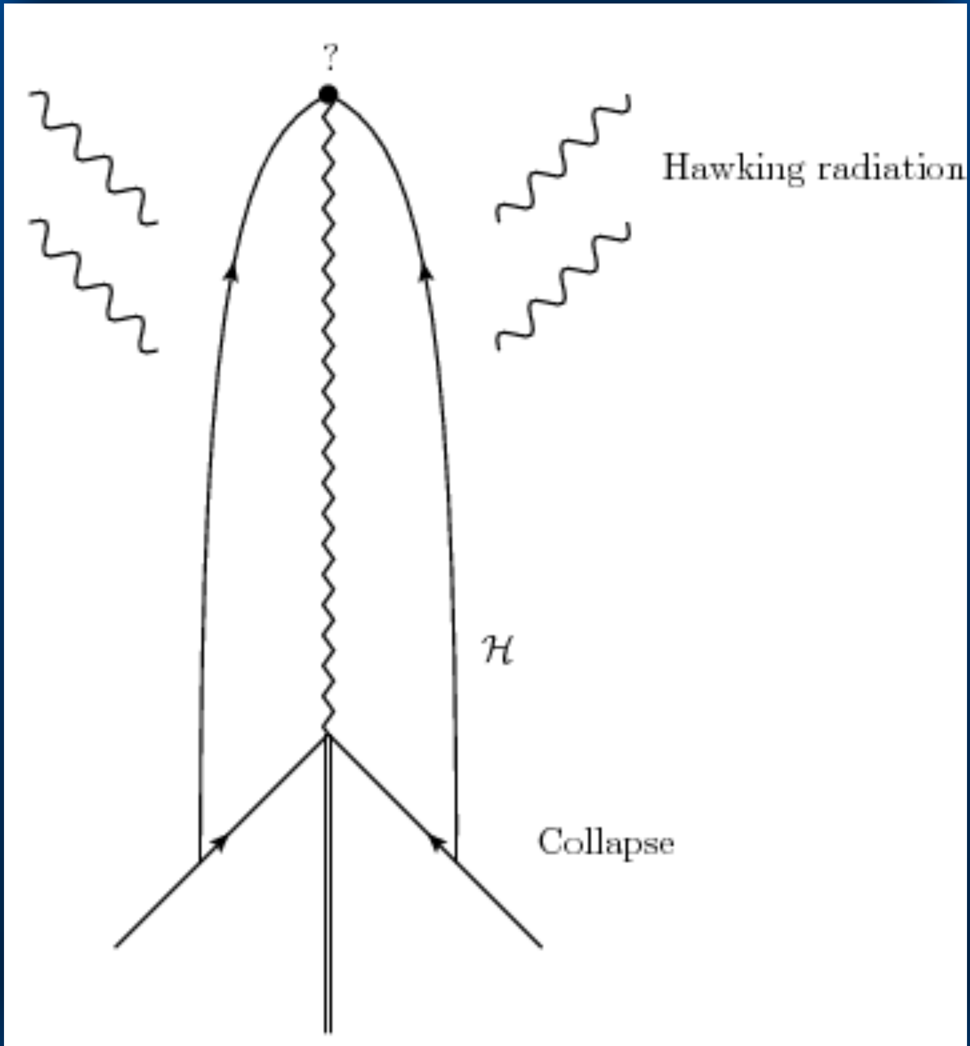


Figure 10.6: Power spectra from a non-rotating black hole. The curves plot the contributions of four neutrino species (1), photons (2), and gravitons (3), and the aggregate spectrum (4). For the sake of comparison, the black body radiation spectrum of these particles with cross-section $27\pi M^2$ is given (curve 5) [data from Page (1976a)].


?
Hawking radiation
$\mathcal{H}$
Collapse

Heisenberg picture

In the Heisenberg picture of quantum mechanics the state vectors, $|\psi\rangle$, do not change with time, while operators satisfy

$$\frac{dA}{dt} = i[H, A].$$

If H is independent of time:

$$A(t) = U(t)^{-1} A(0)\, U(t), \qquad U(t) \equiv e^{-itH}.$$

$U(t)$ is a unitary operator, so $U(t)^{-1} = U(t)^{\dagger} = e^{+itH}$.

Quantum aspects of black holes

To study Hawking radiation we have to estimate the expectation value of the energy-momentum tensor of quantum fields in the vacuum close to the horizon.
v: in, *u*: out

$$ds^2 = C(r)\,\mathrm{d}u\,\mathrm{d}v$$

$$\left.\begin{aligned} u &= t - r^* + R_0^* \\ v &= t + r^* - R_0^* \end{aligned}\right\}$$

$$r^* = \int C^{-1}\,\mathrm{d}r,$$

There is an event horizon at some value of r for which $C=0$ (we assume the spacetime to be non-singular outside the horizon). For example $C = 1 - 2Mr^{-1}$ has an event horizon at $r = 2M$, and models in two dimensions the Schwarzschild black hole. Similarly, $C = 1 - 2Mr^{-1} + e^2r^{-2}$ models the Reissner–Nordstrom black hole.

There will, however, be a nonzero vacuum 'polarization' stress due to spacetime curvature,

$$\langle 0|T_{uu}|0\rangle = \langle 0|T_{vv}|0\rangle = -F_u(C) = (1/192\pi)[2CC'' - C'^2]$$

$$\langle 0|T_{uv}|0\rangle = (1/96\pi)CC'',$$

where the functional F is defined by

$$F_x(y) = \frac{1}{12\pi} y^{\frac{1}{2}} \frac{\partial^2}{\partial x^2}(y^{-\frac{1}{2}}),$$

and a prime denotes differentiation with respect to r.

For a Reissner-Nrodstrom black hole:

$$C(r) = (1 - 2Mr^{-1} + e^2 r^{-2}),$$

$$\langle 0| T_{uu} |0\rangle = \langle 0| T_{vv} |0\rangle = \frac{1}{24\pi}\left[-\frac{M}{r^3} + \frac{3}{2}\frac{M^2}{r^4} + \frac{3e^2}{2r^4} - \frac{3Me^2}{r^5} + \frac{e^4}{r^6} \right]$$

R=2*M*, *e*=0 $\quad \rightarrow \dfrac{\kappa^2}{48\pi},$

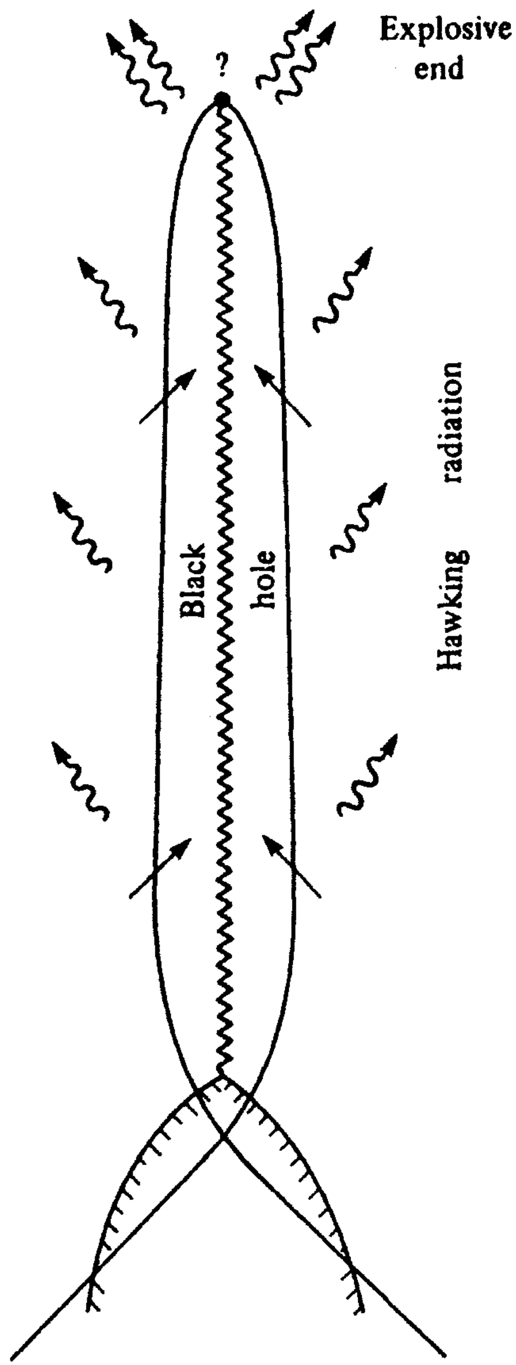
which is precisely the flux expected (in two spacetime dimensions) from a thermal radiator with temperature $T = \kappa/2\pi k_B$

which is the Hawking temperature

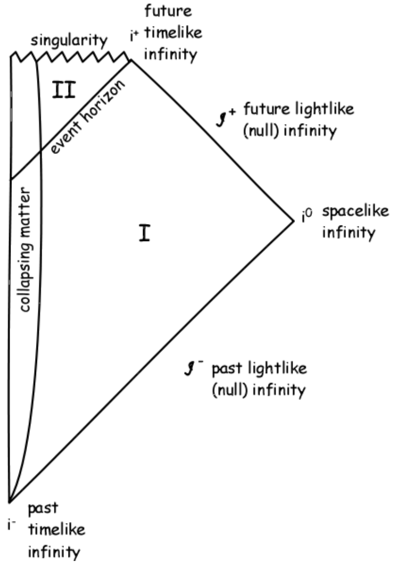
Inspection of $\langle \hat{0} | T_{vv} | \hat{0} \rangle$ at the event horizon shows that it is given by

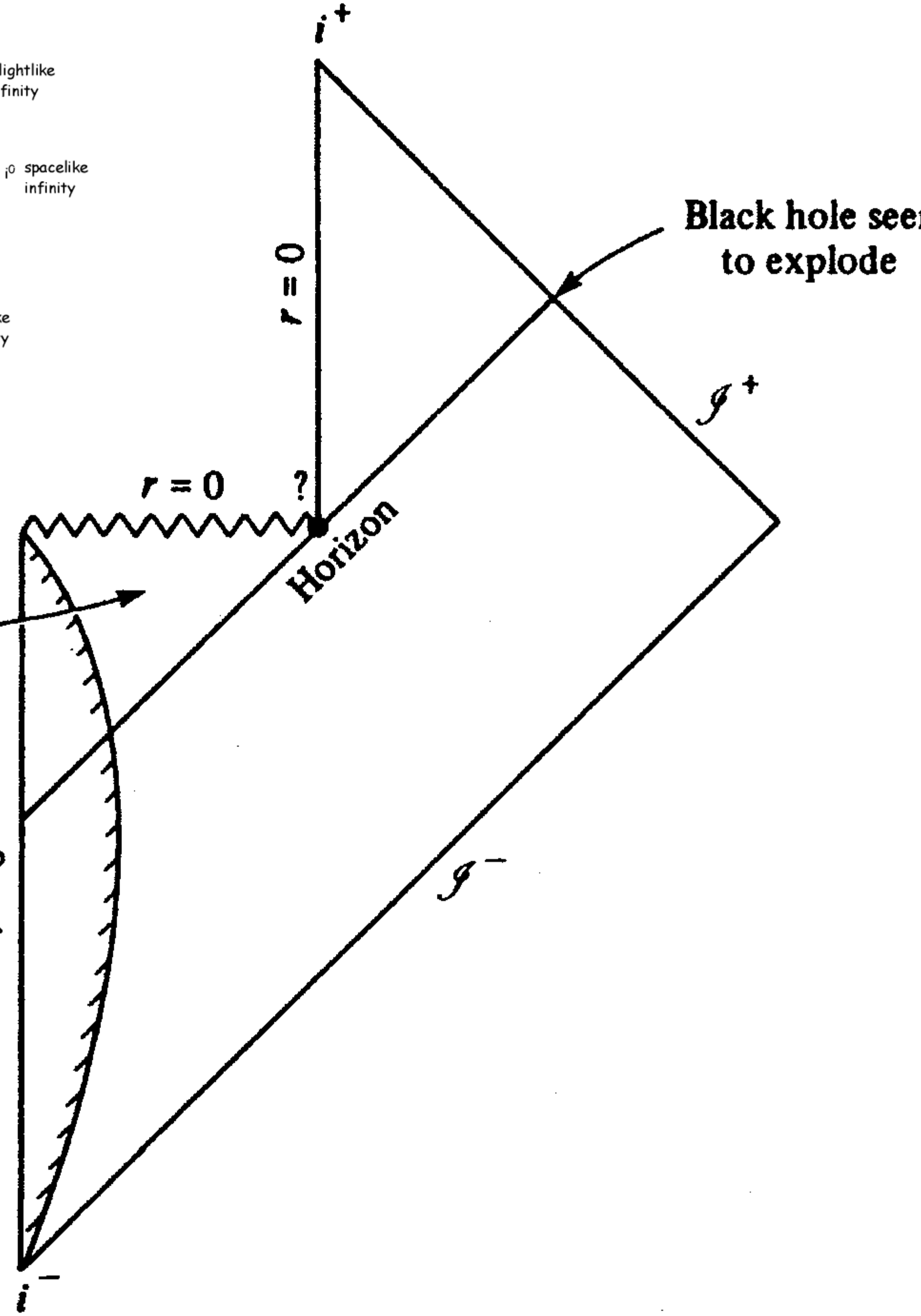
$$-\frac{1}{192\pi}\left(\frac{\partial C}{\partial r}\right)^2\Bigg|_{r=R_h} = -\frac{\kappa^2}{48\pi},$$

which is always negative, and equal to minus the Hawking flux at infinity. This is necessarily true because covariant conservation was built into the construction of $\langle T_{\mu\nu} \rangle$. As $\langle T_{vv} \rangle$ represents a null flux crossing the event horizon, one can see that the steady loss of mass–energy by the Hawking flux is balanced by an equal negative energy flux crossing into the black hole from outside. The hole therefore loses mass, not by emitting quanta, but by absorbing negative energy.



Explosive end
?
Black hole
Hawking radiation


future
timelike
infinity
i^+
singularity
II
event horizon
$\mathscr{I}^+$ future lightlike
(null) infinity
i^0 spacelike
infinity
I
collapsing matter
$\mathscr{I}^-$ past lightlike
(null) infinity
i^- past
timelike
infinity


i^+
$r = 0$
Black hole seen
to explode
$\mathscr{I}^+$
$r = 0$
?
Horizon
Black hole
$r = 0$
$\mathscr{I}^-$
i^-

$$\langle 0_K | N_\sigma | 0_K \rangle = \sum_{n=0}^{\infty} n_\sigma e^{-\beta E_n} \Big/ \sum_{m=0}^{\infty} e^{-\beta E_m}$$

$$= 1/(e^{\beta\sigma} - 1),$$

i.e., a Planck spectrum.

with $E_n = n\omega, \beta = 2\pi/\kappa$

Hawking's work places Bekenstein's conjecture on a firm foundation, and supplies the precise relation (see, for example, DeWitt 1975)

$$\mathscr{S} = \tfrac{1}{4} k_B \mathscr{A}.$$

Hawking's temperature $T = \kappa / 2\pi k_B$

Black hole information paradox

“Is information destroyed by black holes?”

Hawking: Yes (1976), No (2011).

What is information? It is a polysemic word. In ordinary usage ‘information’ is a property of languages. In information theory ‘information’ is a measure of the amount of data transmitted in a signal from a transmisor to a receiver and decoder.

Hence, there is no “law of information conservation”. Actually, information is usually lost when a signal is sent from the sender to the receiver.

Some authors confuse “information” with “entropy”, which is a thermodynamic concept.

"Entropic paradox": the entropy of black holes decreases when they evaporate.

The Second Law requires that the entropy increases or is at a maximum only for closed systems. BHs interact with their environment and are open systems.

"Paradox of predictability": GR cannot predict the future evolution of the system after evaporation despite there are no classic Cauchy horizons.

Yes: GR is *incomplete* (Penrose theorems)

BH-QM paradox

Take a quantum system in a pure state and throw it into a black hole. Wait until the hole has evaporated enough to return to its mass previous to throwing anything in. What we start with is a pure state and a black hole of mass M. What we end up with is a thermal state and a black hole of mass M. We have found a process (apparently) that converts a pure state into a thermal state. But, a thermal state is a MIXED state (described quantum mechanically by a density matrix rather than a wave function). We took a state described by a set of eigenvalues and coefficients, a large set of numbers, and transformed it into a state described by temperature, one number.

In technical jargon, the black hole has performed a non-unitary transformation on the state of system.

There are several possible solutions to this problem :

1.Quantum mechanics is not longer valid inside the BH (Hawking, then).

2.Relativity is no longer valid.

3.Hawking radiation does not exist.

4. Black holes do not exist (Hawking, before dying).

5.The evolution of the quantum system is no unitary and there is no problem (system).

Black hole interiors

The most relevant feature of a Schwarzschild black hole interior is that the roles of space and time are exchanged: the space radial direction becomes time, and time becomes a space direction. Inside a spherical black hole, the radial coordinate becomes *time-like*: changes occur in a prefer direction, i.e. toward the space-time singularity.

This means that the black hole interior is essentially dynamic.

Black hole interiors

If we consider a radially infalling test particle:

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2.$$

The structure of the light cones is defined by the condition $ds = 0$. Writing r_{Schw} once again for the Schwarzschild radius, we get:

$$\left(1 - \frac{r_{\mathrm{Schw}}}{r}\right) c^2 dt^2 - \left(1 - \frac{r_{\mathrm{Schw}}}{r}\right)^{-1} dr^2 = 0. \qquad (251)$$

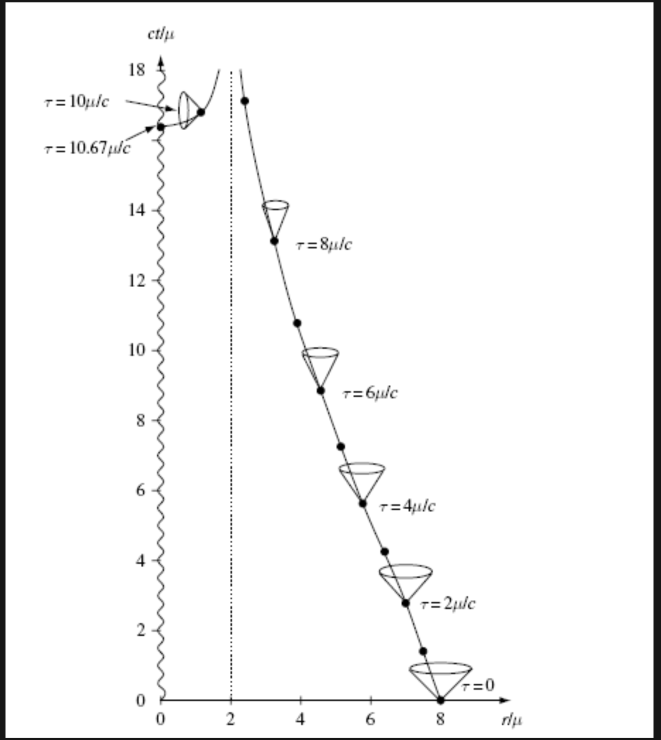
If we consider the interior of the black hole, $r < r_{\mathrm{Schw}}$. Then,

$$\left(1 - \frac{r_{\mathrm{Schw}}}{r}\right)^{-1} dr^2 - \left(1 - \frac{r_{\mathrm{Schw}}}{r}\right) c^2 dt^2 = 0. \qquad (252)$$

Black hole interiors

$$\frac{dr}{dt} = \mp c\left|1 - \frac{r_{\mathrm{Schw}}}{r}\right|,$$

The signs of space and time are now exchanged

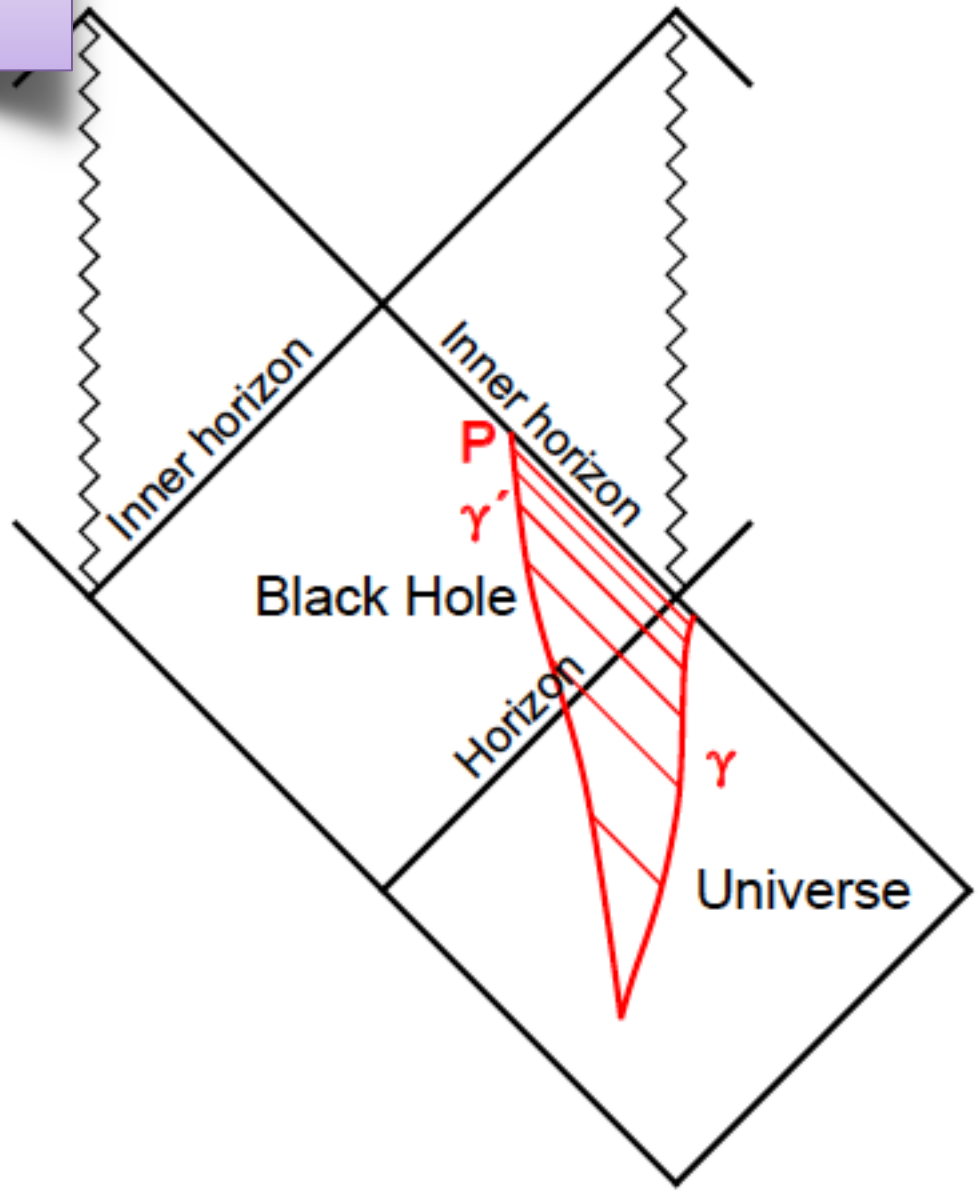


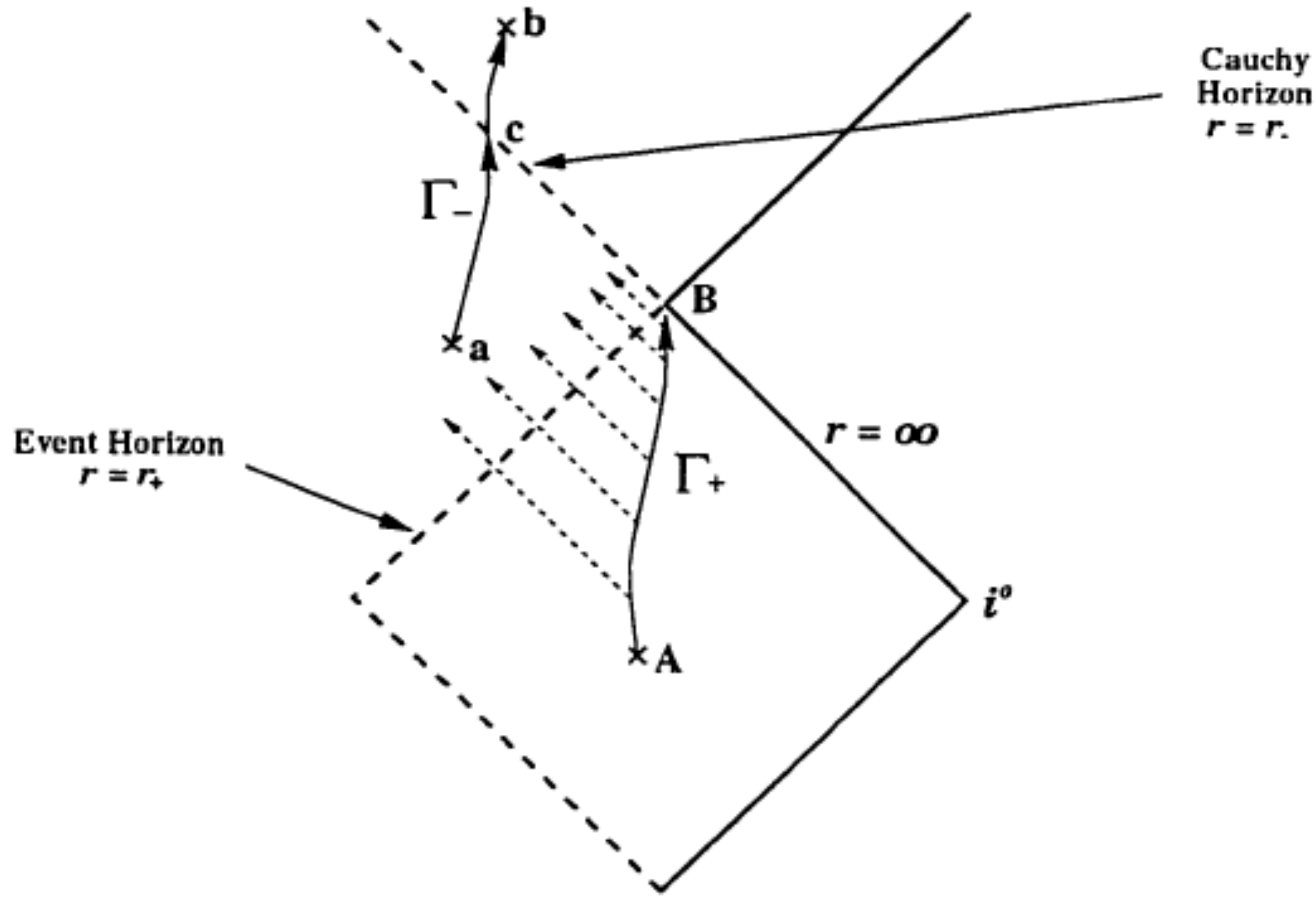
Black hole interiors

In the case of a Kerr black hole, between the two horizons space and time also exchange roles as it happens with the Schwarzschild interior black hole space-time. The radial dimension of space moves inexorably inward to the second horizon, that it is also a Cauchy horizon, i.e. a null hyper-surface beyond which predictability breaks down.

The Kerr solution predicts a second reversal so that one can avoid the ring singularity and achieve to orbit safely. In this strange region inside the Cauchy horizon the observer can, by selecting a particular orbit around the ring singularity, travel backwards in time and meet herself, i.e. there are closed time-like curves.

Bifurcated supertasks for hypercomputation?





b
c
Cauchy
Horizon
r = r-
Γ-
B
a
r = oo
Event Horizon
r = r+
Γ+
i°
A

Israel & Poisson (1990) showed:

$$m_{in}(v) \sim e^{\kappa v} \frac{d}{dv} \delta m(v) ,$$

where v is the advanced time coordinate, k the surface gravity of the Cauchy horizon and δm the equivalent mass of the inflow. The exponential always dominate, producing a divergence when v goes to ∞

Black hole interiors: singularities

- A space-time is said to be *singular* if the manifold M that represents it is *incomplete*.

- A manifold is incomplete if it contains at least one *inextensible* curve.

- A curve $\gamma : [0, a) \rightarrow M$ is inextensible if there is no point p in M such that $\gamma(s) \rightarrow p$ as $a \rightarrow s$, i.e. γ has no endpoint in M. A given space-time (M, g_{ab}) has an *extension* if there is an isometric embedding θ: $M \rightarrow M$, where (M, g_{ab}) is a space-time and θ is onto a proper subset of M.

Black hole interiors: singularities

A space-time is *singular* if it contains a curve γ that is inextensible in the sense given above. Singular space-times are said to contain singularities, but this is an abuse of language: singularities are not "things" in spacetime, but a pathological feature of the theory. Actually, <u>"singularities" cannot exist in space-time by definition.</u>

Black hole interiors: singularities

An essential singularity occurs when $g_{tt} \to \infty$

Essential singularities (i.e. essentially singular space-time models):

- Space-like: unavoidable
- Time-like: avoidable
- Scalar: all scalars diverge
- Non-scalar: some scalars remain finite.

Black hole interiors: singularities

An essential or true singularity should not be interpreted as a representation of a physical object of infinite density, infinite pressure, etc. Since the singularity does not belong to the manifold that represents space-time in General Relativity, it simply cannot be described or represented in the framework of such a theory.

General Relativity is ***incomplete*** in the sense that it cannot provide a full description of the gravitational behavior of all physical systems. True singularities are not within the range of values of the bound variables of the theory: they do not belong to the ontology of a world that can be described with 4-dimensional differential manifolds.

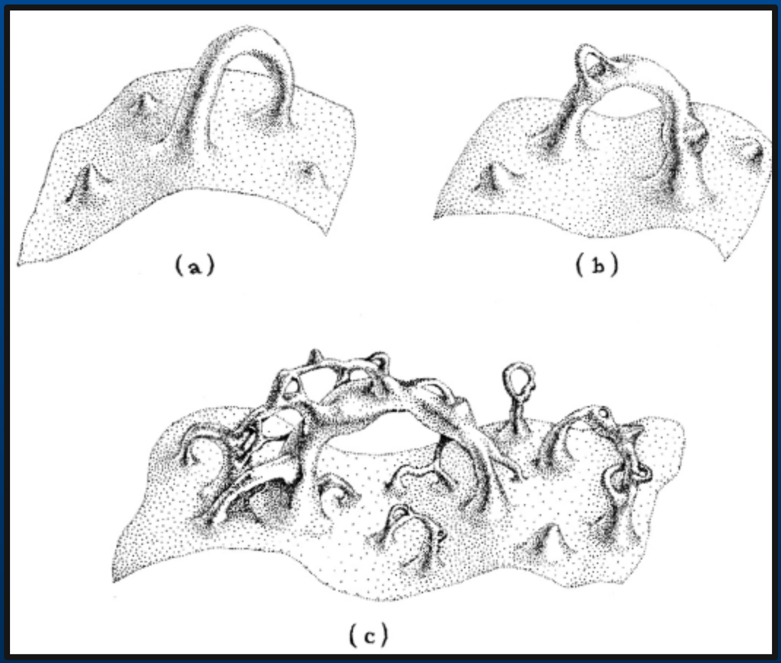
Black hole interiors: singularities

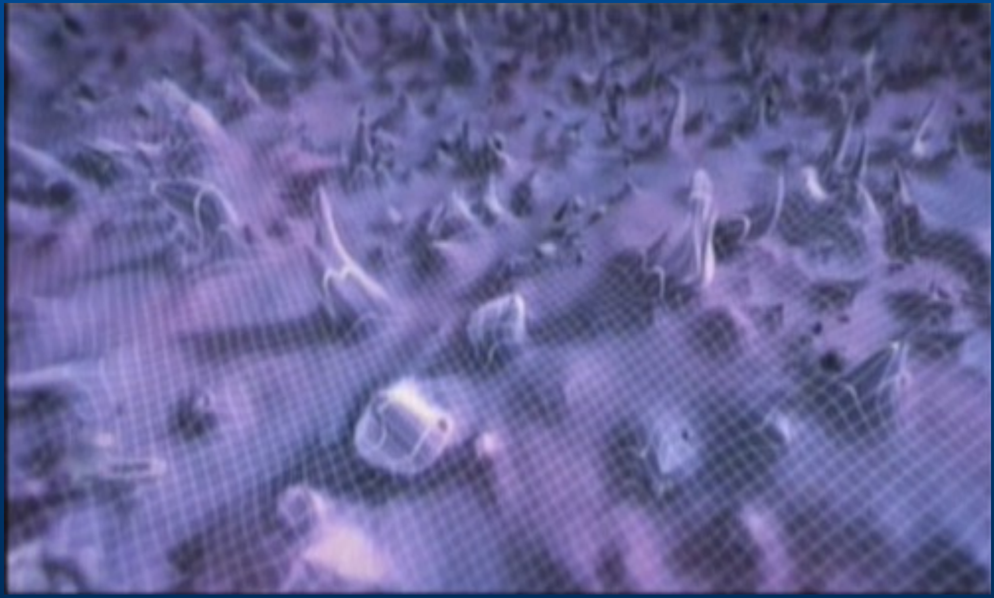
An essential singularity in solutions of the Einstein field equations is one of two things:

1. A situation where matter is forced to be compressed to a point (a space-like singularity).
2. A situation where certain light rays come from a region with infinite curvature (time-like singularity).

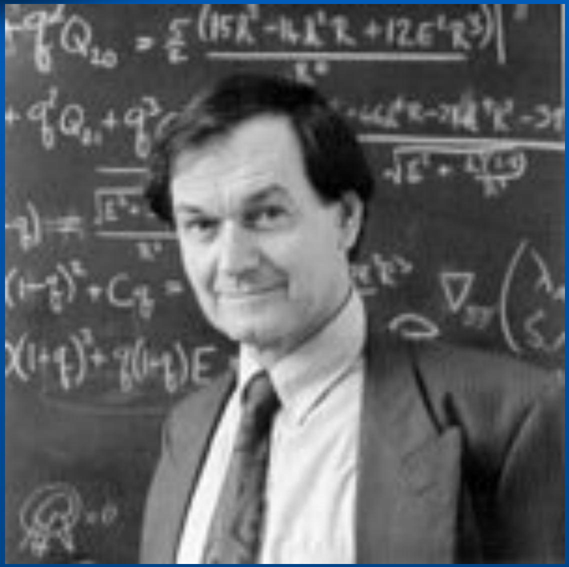
Space-like singularities are a feature of non-rotating uncharged black-holes, whereas time-like singularities are those that occur in charged or rotating black hole exact solutions, where time-like or null curves can always avoid hitting the singularities.

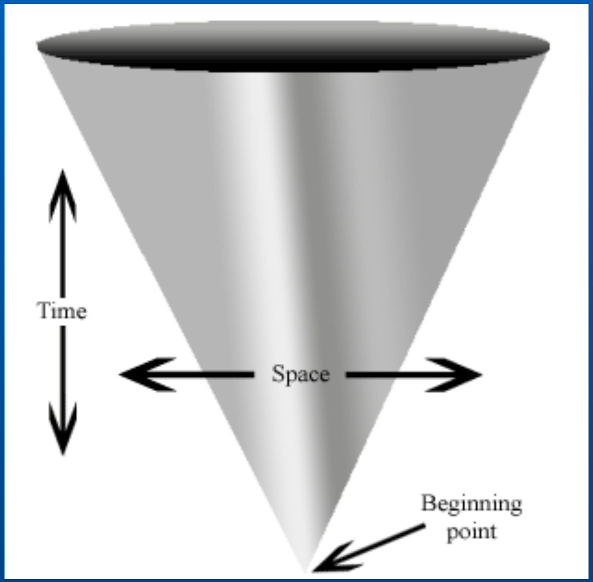
What is referred to as a singularity does not belong to classical space-time. Matter is compressed to such a point that its effects on space-time cannot be described by General Relativity. At such small scales and high densities, relations among things should be described in a quantum mechanical way. <u>Since even in the standard quantum theory time appears as a continuum variable, a new approach is necessary.</u>





Black hole interiors: censorship





Space-time singularities are expected to be covered by horizons. Although formation mechanisms for naked singularities have been proposed, the following conjecture is usually considered valid:

- Cosmic Censorship Conjecture (Roger Penrose): Singularities are always hidden behind event horizons. (space-like)

We emphasize that this conjecture is not proved in General Relativity and hence it has not the strength of a theorem of the theory.

Singularity theorems

Theorem. Let $(M,\ g_{ab})$ a time-oriented space-time satisfying the following conditions:

1. $R_{ab}V^aV^b \geq 0$ for any non space-like V^a.
2. There exists a compact space-like hypersurface $\Sigma \subset M$ without edge.
3. The unit normals to Σ are everywhere converging (or diverging).

Then, $(M,\ g_{ab})$ is time-like geodesically incomplete.

Although singularity theorems apply to spherically symmetric black holes, they do not seem to apply to the Universe as a whole.

The rate of change of the volume expansion as the time-like geodesic curves in the congruence are moved along is given by the Raychaudhuri (1955) equation:

$$\frac{d\theta}{d\tau} = -R_{ab}V^aV^b - \frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab},$$

or

$$\frac{d\theta}{d\tau} = -R_{ab}V^aV^b - \frac{1}{3}\theta^2 - 2\sigma^2 + 2\omega^2.$$

We can use now Einstein's field equations to relate the congruence with the space-time curvature:

$$R_{ab}V^aV^b = \kappa\left[T_{ab}V^aV^b + \frac{1}{2}T\right].$$

The term $T_{ab}V^aV^b$ represents the energy density measured by a time-like observer with unit tangent for velocity V^a. The weak energy condition then states that:

$$T_{ab}V^aV^b \geq 0. \quad \text{WEC}$$

A stronger condition is:

$$T_{ab}V^aV^b + \frac{1}{2}T \geq 0. \quad \text{SEC}$$

Notice that this condition implies,

$$R_{ab}V^aV^b \geq 0$$

We see then that the conditions of the Hawking-Penrose theorem imply that the focusing of the congruence yields:

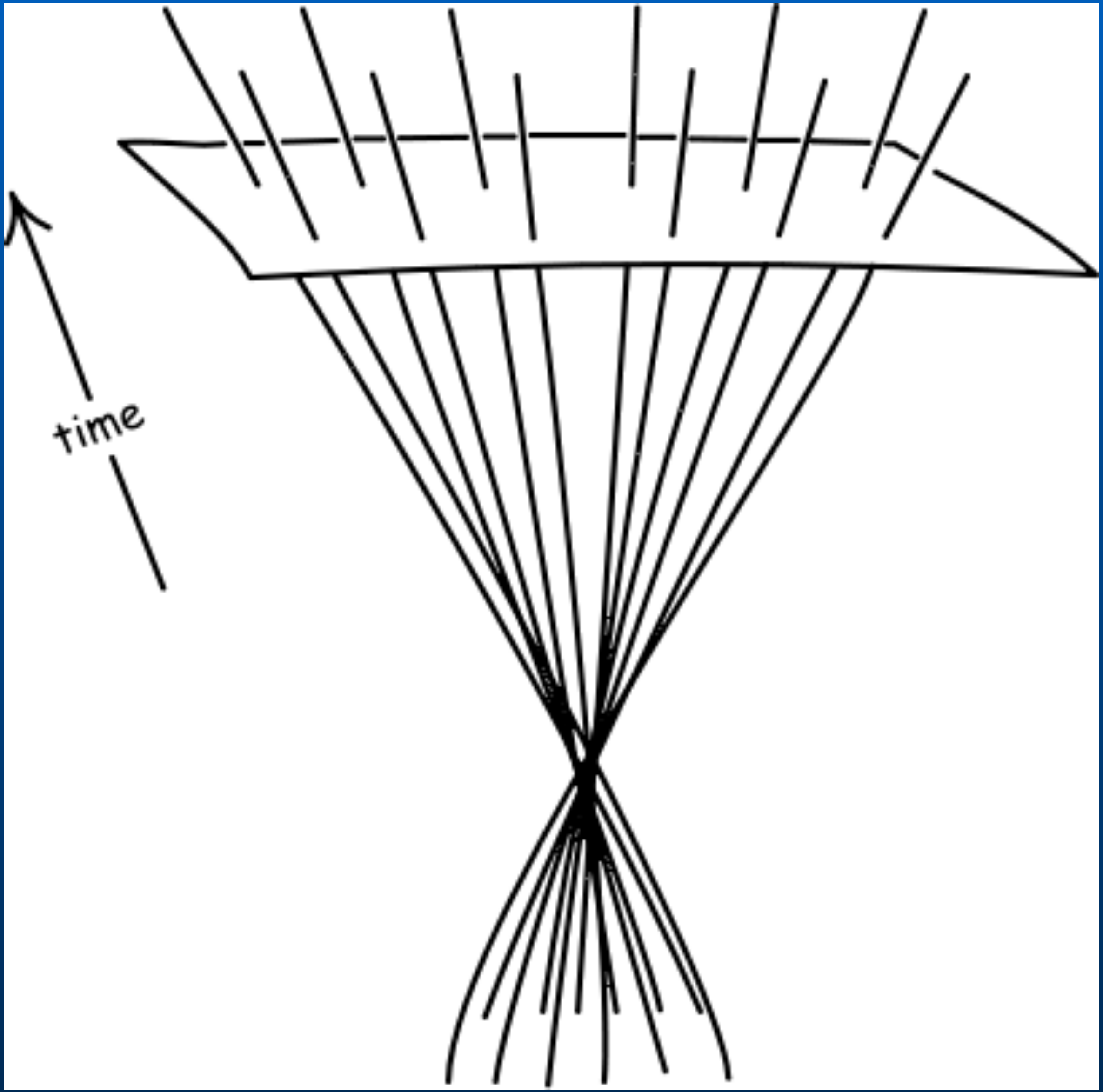
$$\frac{d\theta}{d\tau} \leq -\frac{\theta^2}{3}, \tag{3.44}$$

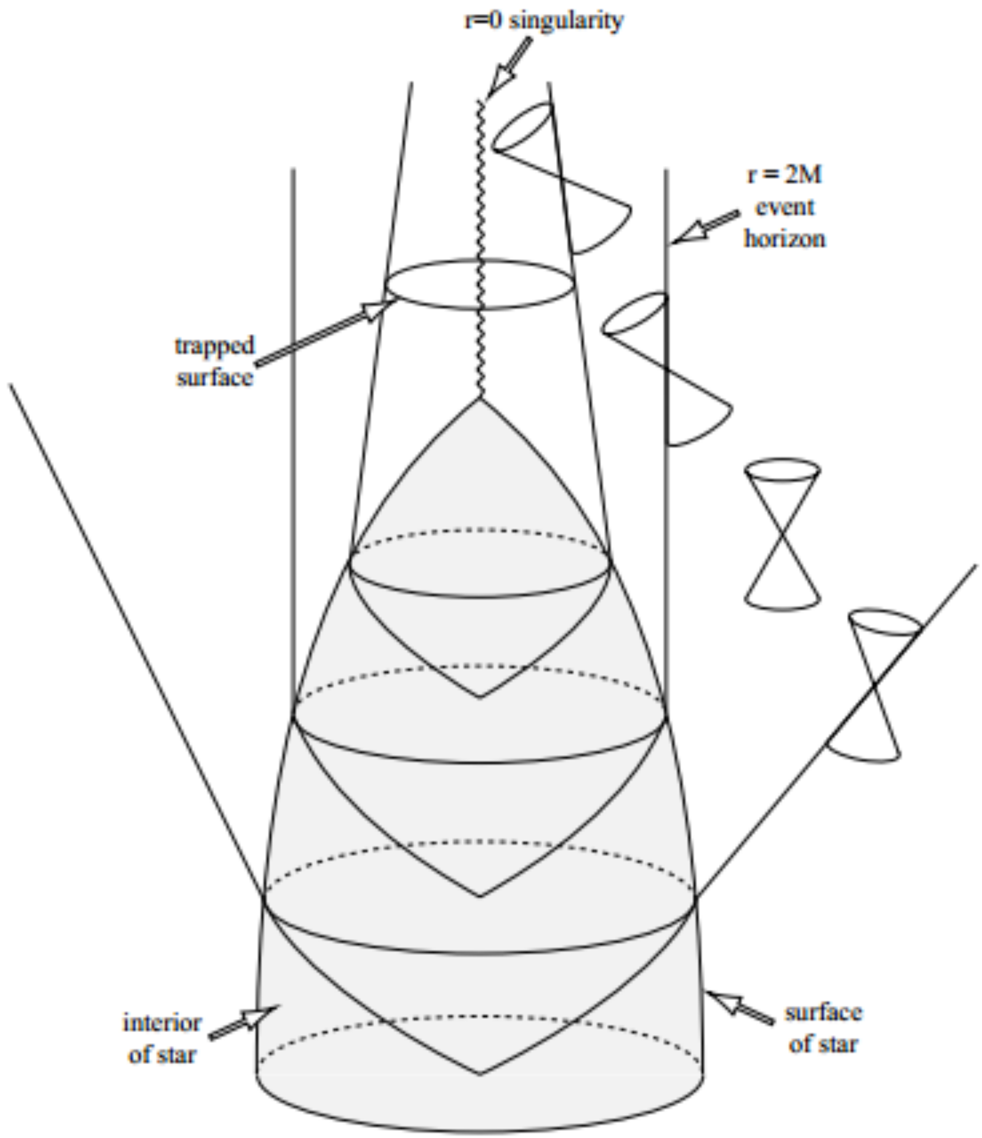
where we have used that both the shear and the rotation vanishes. Equation (3.44) indicates that the volume expansion of the congruence must be necessarily decreasing along the time-like geodesic. Integrating, we get:

$$\frac{1}{\theta} \geq \frac{1}{\theta_0} + \frac{\tau}{3},$$

where θ_0 is the initial value of the expansion. Then, $\theta \to -\infty$ in a finite proper time $\tau \leq 3/|\theta_0|$. This means that once a convergence occurs in a congruence of time-like geodesics, a caustic must develop in the space-time model. The non space-like geodesics are in such a case inextendible.

Singularity theorems




r=0 singularity
r = 2M
event
horizon
trapped
surface
interior
of star
surface
of star

Found Sci (2013) 18:297–306
DOI 10.1007/s10699-012-9309-4

Adversus Singularitates: The Ontology of Space–Time Singularities

Gustavo E. Romero

Published online: 31 October 2012

To interpret the singularity theorems as theorems about the existence of certain space–time models is wrong. Using elementary second order logic is trivial to show that there cannot be non-predicable objects (singularities) in the theory. If there were a non-predicable object in the theory,

$$(\exists x)_E \ (\forall P) \sim Px, \tag{2}$$

where the quantification over properties in unrestricted. The existential quantification $(\exists x)_E$, on the other hand, means

$$(\exists x)_E \equiv (\exists x) \wedge (x \in E).$$

Let us call P_1 the property '$x \in E$'. Then, formula (2) reads:

$$(\exists x)(\forall P)(\sim Px \ \wedge P_1 x), \tag{3}$$

which is a contradiction, i.e. it is false for any value of x.

We conclude that there are no singularities nor singular space–times. There is just a theory with a restricted range of applicability.

Additional: Quantum field theory in curve space-time and quantum black hole effects.

Conventions:

$[A,B]$	$AB - BA$
$\{A,B\}$	$AB + BA$
$a_{(\mu,\nu)}$	$\frac{1}{2}(a_{\mu,\nu} + a_{\nu,\mu})$

1. Quantum mechanics

In quantum mechanics pure quantum states of quantum systems correspond to vectors in a Hilbert space, while each property (such as the energy or momentum of the system) is associated with a mathematical operator. The operator serves as a linear function which acts on the states of the system. The eigenvalues of the operator correspond to the possible values of the property.

The pure states correspond to vectors of norm 1. Thus the set of all pure states corresponds to the unit sphere in the Hilbert space.

A mixed quantum state corresponds to a probabilistic mixture of pure states; however, different distributions of pure states can generate equivalent (i.e., physically indistinguishable) mixed states. Mixed states are described by so-called density matrices. A pure state can also be recast as a density matrix; in this way, pure states can be represented as a subset of the more general mixed states.

The Hilbert space is a phase complex space of infinite dimensions. <u>It is not a physical space-time.</u>

Heisenberg picture

In the Heisenberg picture of quantum mechanics the state vectors, $|\psi\rangle$, do not change with time, while operators satisfy

$$\frac{dA}{dt} = i[H, A].$$

If H is independent of time:

$$A(t) = U(t)^{-1} A(0)\, U(t), \qquad U(t) \equiv e^{-itH}.$$

$U(t)$ is a unitary operator, so $U(t)^{-1} = U(t)^{\dagger} = e^{+itH}$.

$$\langle \Psi, \Phi \rangle \equiv \langle \Psi | \Phi \rangle$$

$$\langle \Psi, \Phi \rangle = \int_{-\infty}^{\infty} \Psi(x)^* \Phi(x)\, dx.$$

$\langle \Psi | x(t) | \Psi \rangle$ and $\langle \Psi | p(t) | \Psi \rangle$ are the expectation values of the position and momentum if those quantities were to be measured at time t with the particle in the state Ψ. The state itself is a time-independent concept (at least so long as the system evolves under its internal dynamics, without interaction with external agencies). The state Ψ is an abstract object in a Hilbert space, $\mathcal{H}$. It is *represented* by a function $\Psi(x) \in \mathcal{L}^2$; this representation has been arbitrarily chosen to be the one which gives directly the probability density for position measurements at $t = 0$.

The canonical commutation relation is the fundamental relation between canonical conjugate quantities (quantities which are related by definition such that one is the Fourier transform of another). For example,

$$[\hat{x}, \hat{p}_x] = i\hbar$$

In general, position and momentum are vectors of operators and their commutation relation between different components of position and momentum can be expressed as

$$[\hat{r}_i, \hat{p}_j] = i\hbar\delta_{ij}.$$

(Algebra structure)
The structure of $\tilde{\mathscr{G}}$, Lie algebra of $\tilde{G}$ is

$$[\hat{J}_i, \hat{J}_j] = i\hbar\varepsilon_{ijk}\hat{J}_k, \qquad [\hat{J}_i, \hat{K}_j] = i\hbar\varepsilon_{ijk}\hat{K}_k, \qquad [\hat{J}_i, \hat{P}_j] = i\hbar\varepsilon_{ijk}\hat{P}_k$$

$$[\hat{K}_i, \hat{H}] = i\hbar\hat{P}_i, \qquad [\hat{K}_i, \hat{P}_j] = i\hbar\,\delta_{ij}\hat{M}$$

$$[\hat{J}_i, \hat{H}] = 0, \qquad [\hat{K}_i, \hat{K}_j] = 0, \qquad [\hat{P}_i, \hat{P}_j] = 0, \qquad [\hat{P}_j, \hat{H}] = 0$$

$$[\hat{J}_i, \hat{M}] = 0, \qquad [\hat{K}_i, \hat{M}] = 0, \qquad [\hat{P}_i, \hat{M}] = 0, \qquad [\hat{H}, \hat{M}] = 0$$

where $\hat{M}$ is an element of the Lie algebra of a one-parameter subgroup (which is used to extend $\bar{G}$).
G is the Galilei group.
$\hat{H}$ is the time-translations generator.

Interpretation

(Probability densities)

$(\forall \langle \sigma, \bar{\sigma} \rangle \in \Sigma \times \bar{\Sigma}) \wedge (\forall \hat{A} \in A \ni \hat{A} \triangleq \mathscr{A}, \quad \mathscr{A} \in \mathscr{P}) \wedge (\forall\ |a\rangle \in \mathscr{H} \ni \hat{A}\ |a\rangle = a\ |a\rangle) \wedge (\forall\ |\psi\rangle \in \Psi \subset \mathscr{H}$ that corresponds to the state of σ when it is influenced by $\bar{\sigma}$):

$\langle \psi | a \rangle \langle a | \psi \rangle \equiv$ probability density for the property $\mathscr{A}$ when σ is associated to $\bar{\sigma}$ (i.e., $\int_{a_1}^{a_2} \langle \psi | a \rangle \langle a | \psi \rangle \, da$ is the probability for σ to have an $\mathscr{A}$-value in $[a_1, a_2]$). (SA)

Interpretation

$\forall \langle \sigma, \bar{\sigma} \rangle \in \Sigma \times \bar{\Sigma}$, eiv $\hat{H} = E$ represents the energy value of σ when it is influenced by $\bar{\sigma}$. (SA)

$\hat{P}_i$ is the generator of spatial translations on the Cartesian coordinate axis X_i.

$\forall \langle \sigma, \bar{\sigma} \rangle \in \Sigma \times \bar{\Sigma}$, eiv $\hat{P}_i = p_i$ represents the i component of the linear momentum of σ. (SA)

$\hat{J}_i$ is the generator of spatial rotations around the Cartesian coordinate axis X_i.

$\forall \langle \sigma, \bar{\sigma} \rangle \in \Sigma \times \bar{\Sigma}$, eiv $\hat{J}_i = j_i$ represents the i component of the angular momentum of σ. (SA)

Interpretation

$\hat{K}_i$ is the generator of pure transformations of Galilei on the axis X_i.
$\hat{M}$ has a discrete spectrum of real and positive eigenvalues.
$\forall \langle \sigma, \hat{\sigma} \rangle \in \Sigma \times \bar{\bar{\Sigma}}$, eiv $\hat{M} = \mu$ represents the mass of σ. (SA)
$\forall \langle \sigma, \hat{\sigma} \rangle \in \Sigma \times \bar{\bar{\Sigma}}$, if $\hat{X}_i =_{\mathrm{Df}} (1/\mu)\hat{K}_i$, then eiv $\hat{X}_i = x_i$ represents the i component of the position of σ. (SA)

$$\Delta x \equiv \left\langle (x - \langle x \rangle)^2 \right\rangle^{\frac{1}{2}}$$

(Heisenberg's inequalities)
$(\forall \langle \sigma, \bar{\sigma} \rangle \in \Sigma \times \bar{\Sigma}) \wedge (\forall |\psi\rangle \in \mathscr{H}) \wedge (\forall \{\hat{A}, \hat{B}, \hat{C}\} \subset A \ni \hat{A} \triangleq \mathscr{A}, \hat{B} \triangleq \mathscr{B}, \hat{C} \triangleq \mathscr{C}$ with $\{\mathscr{A}, \mathscr{B}, \mathscr{C}\} \subset \mathscr{P})$ if $[\hat{A}, \hat{B}] = i\hat{C} \Rightarrow$

$$(\Delta\hat{A})^2 (\Delta\hat{B})^2 \geqslant |\hat{C}|^2/4$$

If $[\hat{X}_i, \hat{P}_j] = \hbar \partial_{ij} \hat{I}$, then

$$\Delta\hat{X}_i \, \Delta\hat{P}_j \geqslant \hbar/2$$

$$(\forall \hat{H}, \hat{A}] = i\hat{C}):$$

$$\longrightarrow \qquad \Delta \hat{H}\, \tau_A \geqslant \frac{\hbar}{2}$$

with $\tau_A = \Delta \hat{A} / |d\langle \hat{A} \rangle / dt|$.

Schrödinger picture

In the Schrödinger picture of quantum mechanics the state vectors, $|\psi(t)\rangle$, evolve with time according to

$$i\hbar\frac{\partial\left|\Psi\right\rangle}{\partial t}=\widehat{H}\left|\Psi\right\rangle$$

2. Quantum fields in flat space-time

A classical physical field is a system whose properties are defined at each point of space and time. They are represented by mathematical functions defined on the manifold that represents space-time. Formally, they have infinite degrees of freedom.

We can define a field model by a Lagrangian density and get the equations of motions using the associated action.

2. Quantum fields in flat space-time

Let us consider a scalar field in Minkwoski space-time

$$\mathscr{L}(x) = \tfrac{1}{2}(\eta^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - m^2\phi^2)$$

$$S = \int \mathscr{L}(x)\mathrm{d}^n x$$

$$\delta S = 0.$$

$$\Rightarrow \quad (\Box + m^2)\phi = 0$$

$$\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$$

2. Quantum fields in flat space-time

$$\phi(t, \mathbf{x}) = \sum_{\mathbf{k}} [a_{\mathbf{k}} u_{\mathbf{k}}(t, \mathbf{x}) + a_{\mathbf{k}}^{\dagger} u_{\mathbf{k}}^{*}(t, \mathbf{x})].$$

$$-\infty < k_i < \infty, \quad i = 1, \ldots, n-1.$$

$$u_{\mathbf{k}}(t, \mathbf{x}) \propto e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}$$

$$\omega \equiv (k^2 + m^2)^{\frac{1}{2}}$$

$$k \equiv |\mathbf{k}| = \left(\sum_{i=1}^{n-1} k_i^2 \right)^{\frac{1}{2}}$$

2. Quantum fields in flat space-time

$$(u_{\mathbf{k}}, u_{\mathbf{k}'}) = 0, \quad \mathbf{k} \neq \mathbf{k}'.$$

$$(u_{\mathbf{k}}, u_{\mathbf{k}'}) = \delta^{n-1}(\mathbf{k} - \mathbf{k}').$$

$$u_{\mathbf{k}} = [2\omega(2\pi)^{n-1}]^{-\frac{1}{2}} e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}$$

2. Quantum fields in flat space-time

The field is quantised by treating it as an operator and imposing the canonical commutation relations.

$$\left.\begin{aligned}
[\phi(t,\mathbf{x}),\phi(t,\mathbf{x}')] &= 0\\
[\pi(t,\mathbf{x}),\pi(t,\mathbf{x}')] &= 0\\
[\phi(t,\mathbf{x}),\pi(t,\mathbf{x}')] &= \mathrm{i}\delta^{n-1}(\mathbf{x}-\mathbf{x}')
\end{aligned}\right\}$$

where π is the canonically conjugate variable to ϕ defined by

$$\pi = \frac{\partial \mathscr{L}}{\partial(\partial_t\phi)} = \partial_t\phi.$$

2. Quantum fields in flat space-time

The equal time commutation relations for ϕ and π are then equivalent to

$$\left.\begin{aligned}[][a_{\mathbf{k}}, a_{\mathbf{k}'}] &= 0\\ [a^\dagger_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] &= 0\\ [a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] &= \delta_{\mathbf{k}\mathbf{k}'}.\end{aligned}\right\}$$

$$a_{\mathbf{k}}|0\rangle = 0, \quad \forall \mathbf{k}.$$

$$|1_{\mathbf{k}}\rangle = a^\dagger_{\mathbf{k}}|0\rangle.$$

Similarly one may construct many-particle states

$$|1_{\mathbf{k}_1}, 1_{\mathbf{k}_2}, \ldots, 1_{\mathbf{k}_j}\rangle = a^\dagger_{\mathbf{k}_1} a^\dagger_{\mathbf{k}_2} \ldots a^\dagger_{\mathbf{k}_j}|0\rangle,$$

if all $\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_j$ are distinct. If any $a^\dagger_{\mathbf{k}}$ are repeated, then

$$|{}^1n_{\mathbf{k}_1}, {}^2n_{\mathbf{k}_2}, \ldots, {}^jn_{\mathbf{k}_j}\rangle = ({}^1n!\,{}^2n! \ldots {}^jn!)^{-\frac{1}{2}} (a^\dagger_{\mathbf{k}_1})^{{}^1n} (a^\dagger_{\mathbf{k}_2})^{{}^2n} \ldots (a^\dagger_{\mathbf{k}_j})^{{}^jn}|0\rangle,$$

the $n!$ terms being necessary to accommodate the Bose statistics of identical scalar particles. Also

$$a^\dagger_{\mathbf{k}}|n_{\mathbf{k}}\rangle = (n+1)^{\frac{1}{2}}|(n+1)_{\mathbf{k}}\rangle$$

$$a_{\mathbf{k}}|n_{\mathbf{k}}\rangle = n^{\frac{1}{2}}|(n-1)_{\mathbf{k}}\rangle.$$

Fock basis

The basis vectors are normalized according to

$$\langle {}^1n_{\mathbf{k}_1}, {}^2n_{\mathbf{k}_2}, \ldots, {}^rn_{\mathbf{k}_r} | {}^1m_{\mathbf{k}'_1}, {}^2m_{\mathbf{k}'_2}, \ldots, {}^sm_{\mathbf{k}'_s} \rangle$$

$$= \delta_{rs} \sum_{\alpha} \delta_{{}^1n^{\alpha(1)}m} \ldots \delta_{{}^rn^{\alpha(s)}m} \delta_{\mathbf{k}_1\mathbf{k}'_{\alpha(1)}} \ldots \delta_{\mathbf{k}_r\mathbf{k}'_{\alpha(s)}}$$

where the sum is over all permutations α of the integers $1 \ldots s$.

$$T_{\alpha\beta} = \phi_{,\alpha}\phi_{,\beta} - \tfrac{1}{2}\eta_{\alpha\beta}\eta^{\lambda\delta}\phi_{,\lambda}\phi_{,\delta} + \tfrac{1}{2}m^2\phi^2\eta_{\alpha\beta}$$

from which one obtains for the Hamiltonian density

$$T_{tt} = \tfrac{1}{2}\left[(\partial_t\phi)^2 + \sum_{i=1}^{n-1}(\partial_i\phi)^2 + m^2\phi^2\right]$$

and for the momentum density

$$T_{ti} = \partial_t\phi\,\partial_i\phi, \quad i = 1, \ldots, n-1,$$

$$H \equiv \int_t T_{tt} \mathrm{d}^{n-1} x = \frac{1}{2} \sum_{\mathbf{k}} (a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + a_{\mathbf{k}} a^\dagger_{\mathbf{k}}) \omega$$

$$P_i \equiv \int_t T_{ti} \mathrm{d}^{n-1} x = \sum_{\mathbf{k}} a^\dagger_{\mathbf{k}} a_{\mathbf{k}} k_i$$

$$[a_{\mathbf{k}},\ a_{\mathbf{k}}^{\dagger}] = \delta_{\mathbf{k}\mathbf{k}'} \rightarrow a_{\mathbf{k}} a_{\mathbf{k}}^{\dagger} - a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} = 1$$

$$a_{\mathbf{k}} a_{\mathbf{k}}^{\dagger} = a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} + 1$$

$$H = \frac{1}{2} \Sigma_{\mathbf{k}} (a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} + a_{\mathbf{k}} a_{\mathbf{k}}^{\dagger}) \omega = \Sigma_{\mathbf{k}} (a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} + \frac{1}{2}) \omega$$

$$\boxed{H = \sum_{\mathbf{k}} (a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} + \tfrac{1}{2})\omega.}$$

Clearly, both H and P_i commute with the operators

$$\left.\begin{aligned} N_{\mathbf{k}} &\equiv a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} \\ N &\equiv \sum_{\mathbf{k}} N_{\mathbf{k}} \end{aligned}\right\}$$

and

$$[N, H] = [N, P_i] = 0.$$

$$\langle 0|N_{\mathbf{k}}|0\rangle = 0, \quad \forall \mathbf{k}$$

$$\langle {}^1n_{\mathbf{k}_1}, {}^2n_{\mathbf{k}_2}, \ldots, {}^jn_{\mathbf{k}_j}|N_{\mathbf{k}_i}|{}^1n_{\mathbf{k}_1}, {}^2n_{\mathbf{k}_2}, \ldots, {}^jn_{\mathbf{k}_j}\rangle = {}^in.$$

$$\langle|N|\rangle = \sum_i {}^in.$$

Thus, the state $|{}^1n_{\mathbf{k}_1}, {}^2n_{\mathbf{k}_2}, \ldots, {}^jn_{\mathbf{k}_j}\rangle$ is a state containing ${}^1n$ quanta in the mode with momentum $\mathbf{k}_1$, ${}^2n$ quanta in the mode with momentum $\mathbf{k}_2$ and so on.

Thus $a_{\mathbf{k}}$ is referred to as an *annihilation operator* and $a_{\mathbf{k}}^{\dagger}$ as a *creation operator*, for quanta in the mode $\mathbf{k}$.

The problem of the divergences

Special interest attaches to the state $|0\rangle$. This is the no-particle, or *vacuum state*. It carries zero momentum

$$\langle 0|\mathbf{P}|0\rangle = 0,$$

$$\langle 0|H|0\rangle = \langle 0|0\rangle \sum_{\mathbf{k}} \tfrac{1}{2}\omega = \sum_{\mathbf{k}} \tfrac{1}{2}\omega$$

$$\sum_{\mathbf{k}} \tfrac{1}{2}\omega = \tfrac{1}{2}(L/2\pi)^{n-1} \int \omega \, d^{n-1}k$$

This sum is infinite!

This infinity apparently indicates that the vacuum contains an infinite energy density. The trouble comes from the ½ *w* zero-point energy with each simple mode of the scalar field. Since *w* has no upper bound, the zero-point energy can be arbitrarily large. In flat space-time, however, we can re-scale or re-normalize the zero-point energy since only differences in energy are meaningful. This is done by ordering the application of the annihilation and creation operators: all annihilation operators must stand to the right of the creation operators. This is indicated by ::

$$:a_{\mathbf{k}}a_{\mathbf{k}}^{\dagger}: = a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}}$$

whence

$$:H: = \sum_{\mathbf{k}} a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}}\omega$$

and the troublesome $\frac{1}{2}\omega$ term has disappeared.

The problem of the divergences

$$\langle 0|T_{\alpha\beta}|0\rangle = \sum_{\mathbf{k}} T_{\alpha\beta}[u_{\mathbf{k}}, u^*_{\mathbf{k}}],$$

$$\langle {}^1n_{\mathbf{k}_1}, {}^2n_{\mathbf{k}_2}, \ldots, |T_{\alpha\beta}|{}^1n_{\mathbf{k}_1}, {}^2n_{\mathbf{k}_2}, \ldots\rangle = \sum_{\mathbf{k}} T_{\alpha\beta}[u_{\mathbf{k}}, u^*_{\mathbf{k}}] + 2\sum_i {}^in\, T_{\alpha\beta}[u_{\mathbf{k}_i}, u^*_{\mathbf{k}_i}].$$

3. Quantum fields in curved space-time

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu, \quad \mu, \nu = 0, 1, \ldots, (n-1)$$

$$g \equiv |\det g_{\mu\nu}|.$$

Scalar field

$$\mathscr{L}(x) = \tfrac{1}{2}[-g(x)]^{\frac{1}{2}}\{g^{\mu\nu}(x)\phi(x)_{,\mu}\phi(x)_{,\nu} - [m^2 + \xi R(x)]\phi^2(x)\}$$

$$S = \int \mathscr{L}(x)\mathrm{d}^n x$$

$$[\Box_x + m^2 + \xi R(x)]\phi(x) = 0$$

Two values of ξ are of particular interest: the so-called minimally coupled case, $\xi = 0$, and the conformally coupled case

$$\xi = \tfrac{1}{4}[(n-2)/(n-1)] \equiv \xi(n).$$

$$(\phi_1, \phi_2) = -\mathrm{i} \int_\Sigma \phi_1(x) \overleftrightarrow{\partial}_\mu \phi_2^*(x) [-g_\Sigma(x)]^{\frac{1}{2}} \mathrm{d}\Sigma^\mu$$

$\mathrm{d}\Sigma^\mu = n^\mu \, \mathrm{d}\Sigma$, with n^μ a future-directed unit vector

$$\phi(x) = \sum_i [a_i u_i(x) + a_i^\dagger u_i^*(x)].$$

Quantification

$$[a_i, a_j^\dagger] = \delta_{ij}, \text{ etc.}$$

Consider, therefore, a second complete orthonormal set of modes $\bar{u}_j(x)$. The field ϕ may be expanded in this set also

$$\phi(x) = \sum_j [\bar{a}_j \bar{u}_j(x) + \bar{a}_j^\dagger \bar{u}_j^*(x)].$$

This decomposition of ϕ defines a new vacuum state $|\bar{0}\rangle$:

$$\bar{a}_j |\bar{0}\rangle = 0, \quad \forall j$$

and a new Fock space.

As both sets are complete, the new modes $\bar{u}_j$ can be expanded in terms of the old:

$$\bar{u}_j = \sum_i (\alpha_{ji} u_i + \beta_{ji} u_i^*).$$

Conversely

$$u_i = \sum_j (\alpha_{ji}^* \bar{u}_j - \beta_{ji} \bar{u}_j^*).$$

These relations are known as Bogolubov transformations

$$\alpha_{ij} = (\bar{u}_i, u_j), \quad \beta_{ij} = -(\bar{u}_i, u_j^*).$$

$$a_i = \sum_j (\alpha_{ji}\bar{a}_j + \beta^*_{ji}\bar{a}^\dagger_j)$$

$$\bar{a}_j = \sum_i (\alpha^*_{ji}a_i - \beta^*_{ji}a^\dagger_i).$$

$$\sum_k (\alpha_{ik}\alpha^*_{jk} - \beta_{ik}\beta^*_{jk}) = \delta_{ij},$$

$$\sum_k (\alpha_{ik}\beta_{jk} - \beta_{ik}\alpha_{jk}) = 0.$$

$|\bar{0}\rangle$ will not be annihilated by a_i:

$$a_i = \sum_j (\alpha_{ji}\bar{a}_j + \beta^*_{ji}\bar{a}^\dagger_j) \qquad a_i|\bar{0}\rangle = \sum_j \beta^*_{ji}|\bar{1}_j\rangle \neq 0$$

. In fact, the expectation value of the operator $N_i = a_i^\dagger a_i$ for the number of u_i-mode particles in the state $|\bar{0}\rangle$ is

$$\langle\bar{0}|N_i|\bar{0}\rangle = \sum_j |\beta_{ji}|^2,$$

which is to say that the vacuum of the $\bar{u}_j$ modes contains $\Sigma_j|\beta_{ji}|^2$ particles in the u_i mode.

Thus, the two sets of modes u_j and $\bar{u}_k$ share a common vacuum state. If any $\beta_{jk} \neq 0$, the $\bar{u}_k$ will contain a mixture of positive-(u_j) and negative-(u_j^*) frequency modes, and particles will be present.

Part of the reason for the nebulousness of the particle concept is its *global* nature. The modes are defined on the whole of spacetime (or at least a large patch) so that a particular observer's specification of the field mode decomposition, and hence the number operator describing the response of a particle detector carried by him, will depend, for example, on the observer's entire past history. To obtain a more objective probe of the state of a field one must construct locally-defined quantities, such as $\langle \psi | T_{\mu\nu}(x) | \psi \rangle$, which assumes a particular value at the point x of spacetime.

In general there is *no* simple relation between $\langle N_i \rangle$ and the particle number as measured by a detector, even if it is freely falling.

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu},$$

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} + \Lambda_{\mathrm{B}} g_{\mu\nu} = -8\pi G_{\mathrm{B}} \langle T_{\mu\nu} \rangle.$$

$$\frac{2}{(-g)^{\frac{1}{2}}} \frac{\delta S_{\mathrm{m}}}{\delta g^{\mu\nu}} = T_{\mu\nu},$$

All two-dimensional spacetimes are conformally flat:

$$g_{\mu\nu} = C(x)\eta_{\mu\nu}.$$

$$\mathrm{d}s^2 = C(u, v)\,\mathrm{d}u\,\mathrm{d}v,$$

$$\left.\begin{aligned}
\theta_{uu} &= -(1/12\pi)C^{\frac{1}{2}}\partial_u^2 C^{-\frac{1}{2}} \\
\theta_{vv} &= -(1/12\pi)C^{\frac{1}{2}}\partial_v^2 C^{-\frac{1}{2}} \\
\theta_{uv} &= \theta_{vu} = 0.
\end{aligned}\right\}$$

4. Quantum aspects of black holes

To study Hawking radiation we have to estimate the expectation value of the energy-momentum tensor of quantum fields in the vacuum close to the horizon.
v: in, *u*: out

$$ds^2 = C(r)\,du\,dv$$

$$\left.\begin{aligned} u &= t - r^* + R_0^* \\ v &= t + r^* - R_0^* \end{aligned}\right\}$$

$$r^* = \int C^{-1}\,dr,$$

There is an event horizon at some value of r for which $C = 0$ (we assume the spacetime to be non-singular outside the horizon). For example $C = 1 - 2Mr^{-1}$ has an event horizon at $r = 2M$, and models in two dimensions the Schwarzschild black hole. Similarly, $C = 1 - 2Mr^{-1} + e^2 r^{-2}$ models the Reissner–Nordstrom black hole.

There will, however, be a nonzero vacuum 'polarization' stress due to spacetime curvature,

$$\langle 0|T_{uu}|0\rangle = \langle 0|T_{vv}|0\rangle = -F_u(C) = (1/192\pi)[2CC'' - C'^2]$$

$$\langle 0|T_{uv}|0\rangle = (1/96\pi)CC'',$$

where the functional F is defined by

$$F_x(y) = \frac{1}{12\pi} y^{\frac{1}{2}} \frac{\partial^2}{\partial x^2}(y^{-\frac{1}{2}}),$$

and a prime denotes differentiation with respect to r.

For a Reissner-Nrodstrom black hole:

$$C(r) = (1 - 2Mr^{-1} + e^2 r^{-2}),$$

$$\langle 0|T_{uu}|0\rangle = \langle 0|T_{vv}|0\rangle = \frac{1}{24\pi}\left[-\frac{M}{r^3} + \frac{3}{2}\frac{M^2}{r^4} + \frac{3e^2}{2r^4} - \frac{3Me^2}{r^5} + \frac{e^4}{r^6}\right]$$

R=2*M*, *e*=0

$$\rightarrow \frac{\kappa^2}{48\pi},$$

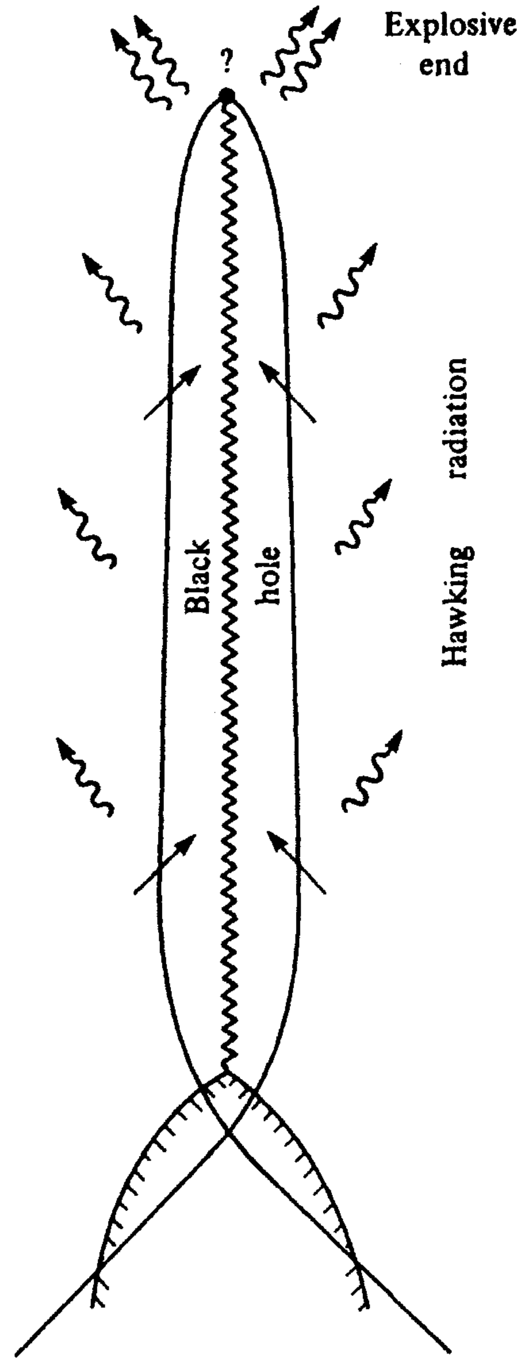
which is precisely the flux expected (in two spacetime dimensions) from a thermal radiator with temperature $T = \kappa/2\pi k_B$

which is the Hawking temperature

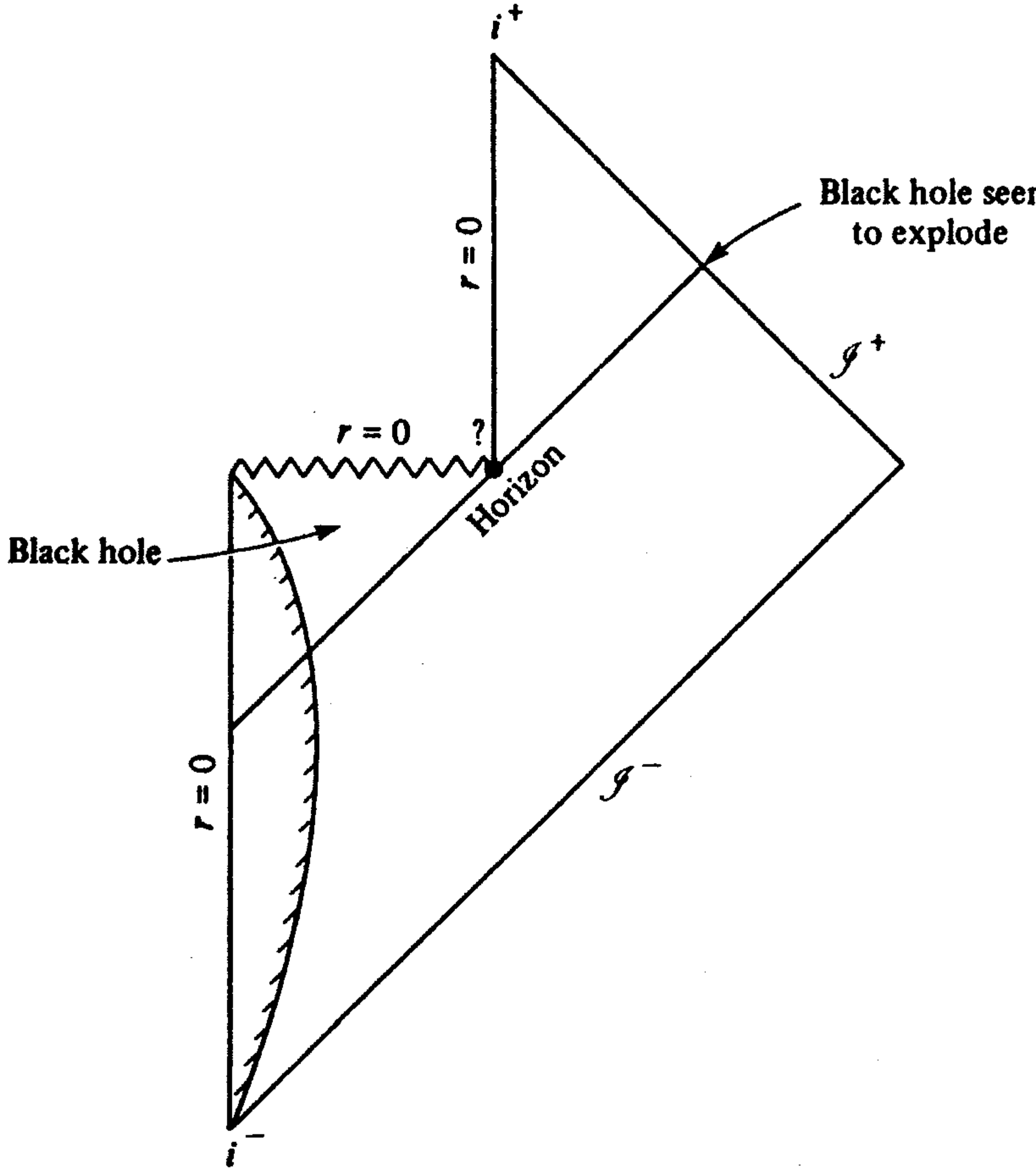
Inspection of $\langle \hat{0} | T_{vv} | \hat{0} \rangle$ at the event horizon shows that it is given by

$$-\frac{1}{192\pi}\left(\frac{\partial C}{\partial r}\right)^2\bigg|_{r=R_h} = -\frac{\kappa^2}{48\pi},$$

which is always negative, and equal to minus the Hawking flux at infinity. This is necessarily true because covariant conservation was built into the construction of $\langle T_{\mu\nu} \rangle$. As $\langle T_{vv} \rangle$ represents a null flux crossing the event horizon, one can see that the steady loss of mass–energy by the Hawking flux is balanced by an equal negative energy flux crossing into the black hole from outside. The hole therefore loses mass, not by emitting quanta, but by absorbing negative energy.



Explosive end
?
Black hole
Hawking radiation



i^+
$r=0$
Black hole seen to explode
$\mathscr{I}^+$
$r=0$
?
Horizon
Black hole
$r=0$
$\mathscr{I}^-$
i^-

$$\langle 0_K | N_\sigma | 0_K \rangle = \sum_{n=0}^{\infty} n_\sigma e^{-\beta E_n} \Big/ \sum_{m=0}^{\infty} e^{-\beta E_m}$$

$$= 1/(e^{\beta\sigma} - 1),$$

i.e., a Planck spectrum.

with $E_n = n\omega, \beta = 2\pi/\kappa$

Hawking's work places Bekenstein's conjecture on a firm foundation, and supplies the precise relation (see, for example, DeWitt 1975)

$$\mathscr{S} = \tfrac{1}{4} k_B \mathscr{A}.$$

Hawking's temperature $T = \kappa/2\pi k_B$

Black hole information paradox

"Is information destroyed by black holes?"

Hawking: Yes (1976), No (2011).

What is information? It is a polysemic word. In ordinary usage 'information' is a property of languages. In information theory 'information' is a measure of the amount of data transmitted in a signal from a transmisor to a receiver and decoder.

Hence, there is no "law of information conservation". Actually, information is usually lost when a signal is sent from the sender to the receiver.

Some authors confuse "information" with "entropy", which is a thermodynamic concept.

“Entropic paradox”: the entropy of black holes decreases when they evaporate.

The Second Law requires that the entropy increases or is at a maximum only for closed systems. BHs interact with their environment and are open systems.

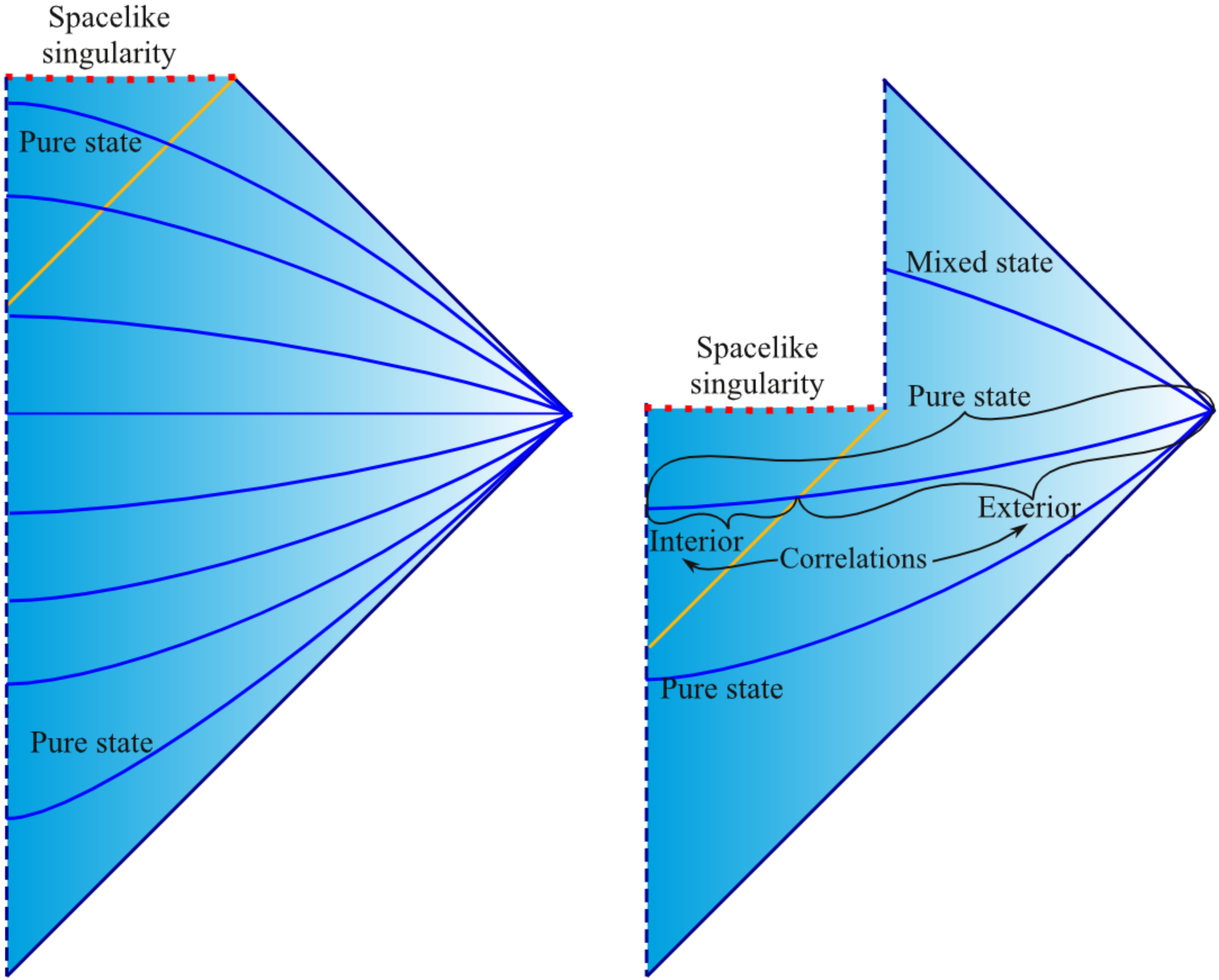
“Paradox of predictability”: GR cannot predict the future evolution of the system after evaporation despite there are no classic Cauchy horizons.

Yes: GR is *incomplete* (Penrose theorems)

BH-QM paradox

Take a quantum system in a pure state and throw it into a black hole. Wait until the hole has evaporated enough to return to its mass previous to throwing anything in. What we start with is a pure state and a black hole of mass M. What we end up with is a thermal state and a black hole of mass M. We have found a process (apparently) that converts a pure state into a thermal state. But, a thermal state is a MIXED state (described quantum mechanically by a density matrix rather than a wave function). We took a state described by a set of eigenvalues and coefficients, a large set of numbers, and transformed it into a state described by temperature, one number.

In technical jargon, the black hole has performed a non-unitary transformation on the state of system.



A. Non-evaporating black hole

B. Evaporating black hole

There are several possible solutions to this problem :

1.Quantum mechanics is not longer valid inside the BH (Hawking, then).

2.Relativity is no longer valid.

3.Hawking radiation does not exist.

4. Black holes do not exist (Hawking, now).

5.The evolution of the quantum system is no unitary and there is no problem (system).